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Robustness of strong stability of semigroups

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Abstract

In this paper we study the preservation of strong stability of strongly continuous semigroups on Hilbert spaces. In particular, we study a situation where the generator of the semigroup has a finite number of spectral points on the imaginary axis and the norm of its resolvent operator is polynomially bounded near these points. We characterize classes of perturbations preserving the strong stability of the semigroup. In addition, we improve recent results on preservation of polynomial stability of a semigroup under perturbations of its generator. Theoretic results are illustrated with an example where we consider the preservation of the strong stability of a multiplication semigroup.

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1. Introduction

The properties of a linear abstract Cauchy problem

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \in X \tag{1}$$

on a Hilbert space X can be studied using the theory of strongly continuous semigroups [8,2]. If the operator $A : \mathcal{D}(A) \subset X \to X$ generates a strongly continuous semigroup $T(t)$ on X, then the initial value problem (1) has a unique solution given by $x(t) = T(t)x_0$ for all $t \geq 0$. In particular,

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the asymptotic behavior and different types of stability of the solutions of (1) can be studied by analyzing the stability properties of the semigroup $T(t)$. The use of semigroups provides a unified approach to developing theory for — for example — classes of linear differential, partial differential, and integral equations that can be written in the form (1).

In this paper we are interested in robustness of the stability properties of the semigroup $T(t)$ in the situation where its infinitesimal generator A is perturbed. It is a well-known fact that the exponential stability of a strongly continuous semigroup is preserved under all bounded perturbations whose operator norms are sufficiently small. However, in a situation where $T(t)$ is not exponentially stable, but merely *strongly stable*, i.e.,

$$\lim_{t\to\infty}\|T(t)x\| = 0, \quad \forall x \in X,$$

no general conditions for the preservation the stability of $T(t)$ are known. On the contrary, it is acknowledged that strong stability may be extremely sensitive to even arbitrarily small perturbations of its infinitesimal generator.

Recently in [13,14] it was shown that a subclass of strongly stable semigroups, the so-called *polynomially stable semigroups*, do indeed possess good robustness properties. The key observation was that in the case of polynomial stability, the size of the perturbation $A + BC$ should not be measured using the regular operator norms $\|B\|$ and $\|C\|$, but instead using graph norms $\|(-A)^\beta B\|$ and $\|(-A^*)^\gamma C^*\|$ for suitable exponents β and γ. The polynomially stable semigroups have a characteristic property that their generators have no spectrum on the imaginary axis $i\mathbb{R}$. Therefore, many of the strongly stable semigroups encountered in applications are beyond the scope of the perturbation results in [13,14]. In this paper we study the robustness properties of semigroups whose generators do have spectrum on the imaginary axis. In particular, we consider a situation where A has a finite number of spectral points on the imaginary axis, and the norm of the resolvent operator of A is polynomially bounded near these points. We show that the semigroups of this type have surprising robustness properties.

The results presented in this paper again demonstrate that for a strongly stable semigroup $T(t)$, the size of the perturbation should not be measured using the regular operator norm, but instead using suitable graph norms related to the generator A. Our main results reveal large and easily characterizable classes perturbations that preserve the strong stability of $T(t)$. The results can be applied, for example, in the study of linear partial differential equations, and in robust control of infinite-dimensional linear systems [15].

To the author's knowledge, robustness properties of strong stability of semigroups with spectrum on the imaginary axis have not been studied previously in the literature. Some results on preservation of strong stability of compact semigroups can be found in [7]. However, any strongly stable compact semigroup is actually exponentially stable [8, Ex. V.1.6(4)].

To illustrate our conditions for the preservation of stability, we begin by stating our main result in a situation where A has a single imaginary spectral point $\sigma(A) \cap i\mathbb{R} = \{0\}$ belonging to the continuous spectrum of A and the perturbing operator is of finite rank. We further assume that there exists $\alpha \geq 1$ such that

$$\sup_{0<|\omega|\leq 1} |\omega|^\alpha \|R(i\omega, A)\| < \infty \quad \text{and} \quad \sup_{|\omega|>1} \|R(i\omega, A)\| < \infty. \tag{2}$$

These assumptions are satisfied, for example, if A generates a strongly stable analytic semigroup with $0 \in \sigma_c(A)$, and $|\lambda|\|R(\lambda, A)\| \leq M$ outside some sector in $\mathbb{C}^-$. Our assumptions on