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Sharpness for C^1 linearization of planar hyperbolic diffeomorphisms ☆

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Abstract

C^1 linearization preserves smooth dynamical behaviors and distinguishes qualitative properties in characteristic directions. Planar hyperbolic diffeomorphisms are the most elementary ones of representatively technical difficulties in the study of C^1 linearization. In the Poincaré domain (both eigenvalues inside the unit circle S^1) a lower bound α_0 was given such that $C^{1,\alpha}$ smoothness with $\alpha_0 < \alpha \leq 1$ admits C^1 linearization. Our first purpose of this paper is to prove the sharpness of α_0 and give a weaker linearization for $\alpha \leq \alpha_0$. In the Siegel domain (one eigenvalue inside S^1 but the other outside S^1) it is known that $C^{1,\alpha}$ smoothness admits C^1 linearization for all $\alpha \in (0, 1]$. The second purpose is to prove that the C^1 linearization is actually a $C^{1,\beta}$ linearization and give sharp estimates for β.

Keywords: C^1 linearization; Hyperbolic diffeomorphism; Invariant manifold; Functional equation; Whitney extension theorem

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1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and $F : X \to X$ be a diffeomorphism such that

$$F(O) = O \quad \text{and} \quad DF(O) = \Lambda, \tag{1.1}$$

where O is the origin and $DF(O)$ is the (Fréchet) differentiation of F at O. Thus Λ is a bounded linear operator defined on X. The local C^r linearization of F is to find a C^r diffeomorphism Φ near O such that the conjugacy equation

$$\Phi \circ F = \Lambda \circ \Phi \tag{1.2}$$

holds. The well known Hartman–Grobman Theorem [13,20] says that C^1 diffeomorphisms on X can be C^0 linearized near hyperbolic fixed points. Here a fixed point of F is said to be *hyperbolic* if Λ has no eigenvalues on the unit circle S^1. In order to preserve more dynamical properties in the procedure of linearization, one expects the solution Φ of Eq. (1.2) to be as regular as possible. This work goes back to Poincaré [19], who investigated analytic linearization for analytic diffeomorphisms. Results on C^r linearization for C^k diffeomorphisms with $1 \le r \le k \le \infty$, initiated by Sternberg [27,28] in 1950s, can be found in [4,5,24].

C^1 linearization is of special interest because it preserves smooth dynamical behaviors and distinguishes characteristic directions of the systems. Its applications can be referred to [3,7] for homoclinic bifurcations, [10] for stability of topological mixing of hyperbolic flows, [15] for Lorenz attractors, [16] for Homoclinic tangencies, and [32] for C^1 iterative roots of mappings. For these reasons great efforts (see e.g. [8,22,23] and references therein) have been made to C^1 linearization of hyperbolic diffeomorphisms in Euclidean spaces and Banach spaces since Hartman's [12] and Belitskii's [4]. Noting some examples (see [17, p. 139] and [26]) of 1-dimensional C^1 hyperbolic mappings which cannot be C^1 linearized, one usually considers $C^{1,\alpha}$ mappings with $\alpha \in (0, 1]$, where $C^{1,\alpha}$ denotes the class of all C^1 mappings F whose derivatives satisfy

$$\sup_{x \ne y} \frac{\|DF(x) - DF(y)\|}{\|x - y\|^{\alpha}} < \infty. \tag{1.3}$$

In spite of some more results on C^1 linearization of 1-dimensional mappings (see Theorems 6.2 and 6.3 in [17]), an important conclusion is that 1-dimensional $C^{1,\alpha}$ hyperbolic mappings can be $C^{1,\alpha}$ linearized for all $\alpha \in (0, 1]$, a corollary of Theorem 6.1 in [17]. More attentions are paid to 2-dimensional or higher-dimensional $C^{1,\alpha}$ mappings. Let F be a planar hyperbolic mapping, $X = \mathbb{R}^2$ and $\lambda := (\lambda_1, \lambda_2)$, where λ_1 and λ_2 are eigenvalues of the linear part Λ. As indicated in [2], there are two cases in discussion: λ lies in the *Poincaré domain* (i.e., either λ_1 and λ_2 both lie inside the unit circle S^1 or both outside S^1); λ lies in the *Siegel domain* (i.e., the complement of the Poincaré domain). In the Poincaré domain, it suffices to discuss in the case $0 < |\lambda_1| \le |\lambda_2| < 1$ because the case of expansion can be reduced to this case by considering the inverse of the mapping. It is known from [6, Corollary 1.3.3] that all $C^{1,\alpha}$ mappings can be $C^{1,\alpha}$ linearized if

$$\alpha > \alpha_1 := \log|\lambda_1| / \log|\lambda_2| - 1. \tag{1.4}$$