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On the behavior of the solution of the dissipative Camassa–Holm equation with the arbitrary dispersion coefficient

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Abstract

In this paper, we consider the dissipative Camassa–Holm equation with arbitrary dispersion coefficient and compactly supported initial data. We demonstrate the simple conditions on the initial data that lead to finite time blow-up of the solution in finite time or guarantee that the solution exists globally. Also, propagation speed for the equation under consideration is investigated.

MSC: 35B44; 35L05; 37K10

Keywords: Dissipative Camassa–Holm equation; Blow-up

1. Introduction

The Camassa–Holm equation

$$u_t - u_{txx} + ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad t > 0,\ x \in \mathbb{R}$$

is a model for wave motion on shallow water which was derived physically by Camassa and Holm by approximating the Hamiltonian for the Euler equations of inviscid fluid dynamics

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directly in the shallow water regime, where $u(t,x)$ represents the fluid's free surface above a flat bottom and k is a dispersive coefficient related to the critical shallow water speed [4,5,7]. The details concerning the hydrodynamical relevance of Camassa–Holm equation were mathematically rigorously described in [7], where, in addition, authors investigate in what sense model under consideration gives us insight into the wave breaking phenomenon. Alternative derivations of Camassa–Holm equation as an equation for geodesic flow on the diffeomorphism group of the circle were presented by Constantin and Kolev [6]. The equation has bi-Hamiltonian structure [14] and is completely integrable [1,2,5,8,9].

A lot of satisfactory results have been obtained for this shallow water equation during recent years. Local well-posedness for the initial datum $u_0(x) \in H^s$ with $s > 3/2$ was proved by several authors (see [11,18,20]). For the initial data with lower regularity, we refer to Molinet's paper [21] and also the recent paper [3]. Moreover, wave breaking for a large class of initial data has been established in [10–12,18,19,23,24] and the recent paper [17], where, in particular, new and direct proof for the result from [19] on the necessary and sufficient condition for wave breaking was presented. It is also worth mentioning that from the point of view of theory of water waves the fact that solutions that originate from smooth localized initial data can develop singularities only in the form of breaking waves, as proved in the paper [10], is especially interesting. In [16], among others, authors showed (for $k = 0$) the infinite propagation speed for the Camassa–Holm equation in the sense that a strong solution of the Cauchy problem with compact initial profile cannot be compactly supported at any later time unless it is the zero solution, which is an improvement of a first result in this direction obtained in [13] (also, see [15]). The wide range of problems for CH equation with non-zero dispersion coefficient was considered in [18,22,25]. In particular, certain conditions on the initial datum to guarantee that the corresponding solution exists globally or blows up in finite time were established.

In the present paper we consider Cauchy problem for the following equation with dissipative term and compactly supported initial data:

$$u_t - u_{xxt} + k(u - u_{xx})_x + 3uu_x + \lambda(u - u_{xx}) = 2u_x u_{xx} + uu_{xxx}, \quad t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \tag{2}$$

where $\operatorname{supp} u_0$ belongs to $[a,b]$, $\lambda > 0$.

Our main aim is to find a simple condition guaranteeing blow-up of the solution by using some properties of the solution generated by initial data. In particular, we use the representation of the solutions on the lines defined by the following problem (see [19,22])

$$q_t = u\big(q(x,t),t\big) + k, \tag{3}$$

$$q(x,0) = x. \tag{4}$$

Precisely, some arguments (see [13,15,16]) used to investigate IPS (the infinite propagation speed) property, help shed additional light on the blow-up mechanism. Also, we will see that there exist some differences between the cases $k, \lambda \neq 0$ and $k, \lambda = 0$ if we speak about the behavior of the solution on the above-mentioned curves.

Our first goal is to prove that "blow-up" phenomena cannot take place under the condition $\|y_0\|_{L^2} < \frac{2\lambda}{3}$.

Theorem 1. *Assume $u_0 \in H^s(\mathbb{R})$, $s \geqslant 3$, is such that the associated potential $y_0 = u_0 - u_{0xx}$ satisfies $\|y_0\|_{L^2} < \frac{2\lambda}{3}$. Then the corresponding solution $u(x,t)$ to Eq. (1) exists globally.*