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Unique continuation principle for the Ostrovsky equation with negative dispersion

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ABSTRACT

In this article we prove that if the difference of two solutions of the Ostrovsky equation with negative dispersion,

$$\partial_t u + \partial_x^3 u - \partial_x u + u\partial_x u = 0,$$

has certain exponential decay for $x > 0$ at two different times, then both solutions are equal.

1. Introduction

In this article we consider the unique continuation principle for the Ostrovsky equation with negative dispersion

$$\partial_t u + \partial_x^3 u - \partial_x^{-1} u + u\partial_x u = 0, \quad x \in \mathbb{R},\ t \in \mathbb{R}. \tag{1.1}$$

Here $\partial_x^{-1} u$ is a spatial antiderivative of u which is defined through the Fourier transform by the multiplier $-i/\xi$. This equation is a perturbation of the well-known Korteweg–de Vries (KdV) equation with the nonlocal term $-\partial_x^{-1} u$ and it was introduced in [11] as a model to describe the propagation of dispersive one-dimensional waves in a rotating frame of reference. Regarding the well-posedness of the Cauchy problem associated to Eq. (1.1), in the context of Sobolev spaces $H^s(\mathbb{R})$, it is known that the problem is locally well-posed for $s > -\frac{3}{4}$ [7] and globally well-posed for $s > -\frac{3}{10}$ [6].

Our purpose is to give sufficient conditions for the difference of two solutions u_1 and u_2 of (1.1) which guarantee that both solutions are identical. The study of continuation properties has been

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considered for a wide variety of dispersive equations. In [13], for a general class of dispersive equations, Saut and Scheurer proved that if a solution $u = u(x,t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, of an equation in such class, vanishes in an open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}$, then it vanishes in all the horizontal components of Ω, that is, in $\mathbb{R}^n \times \{t\colon \exists x \text{ with } (x,t) \in \Omega\}$. Using methods of Complex Analysis, in [1] Bourgain showed that if a solution of the Korteweg–de Vries equation has a common compact support for all times in a nontrivial interval, then the solution is the zero solution. Though weaker than the result in [13], Bourgain's result was easily generalized to other dispersive equations not belonging to the class considered in [13]. (See for example [12].)

In [8], for the KdV equation, Kenig, Ponce, and Vega considered conditions, only at two different times, on the support of a solution u, proving that if this solution is supported in an interval $(-\infty, B]$ at $t = 0$ and $t = 1$, then $u \equiv 0$. A similar result was obtained in [9] for the difference of two solutions u_1 and u_2. This type of conditions, imposed on the supports, has been replaced in subsequent articles by hypotheses on the spatial decay at infinity of the difference $v := u_1 - u_2$, which guarrantee that both solutions are identical. (See [5], and [4] for the KdV and Schrödinger equations, respectively.) This decay is related to the natural decay of the fundamental solution of the equation.

For the Ostrovsky equation with positive dispersion (with "+" sign in the nonlocal term in (1.1)), it was proved in [2], that if for all $a > 0$ the difference $u_1 - u_2$ of two solutions decays, roughly speaking, as $e^{-ax^{3/2}}$, for $x > 0$, then $u_1 \equiv u_2$.

The main result we will establish in this article is the following:

Theorem 1. *Let $u_1, u_2 \in C([0,1]; H^4(\mathbb{R})) \cap C^1([0,1]; L^2(\mathbb{R})) \cap L^\infty([0,1]; L^2((1+x_+)^{2\gamma}\,dx))$, for some $\gamma > 1$, be two solutions of (1.1) and define $v := u_1 - u_2$.*

Suppose that $v(0), v(1) \in L^2(e^{ax_+^{8/5}}\,dx)$ for all $a > 0$. Then $v \equiv 0$. (Here, $x_+ := \frac{1}{2}(x + |x|)$.)

The proof of Theorem 1, which follows the ideas of [5], is obtained by superposing two types of estimates: Carleman type estimates, and a lower estimate. The Carleman estimates express boundedness properties of the inverse of the operator associated to the linear part of the equation in spaces of L^pL^q type with exponential weight. The lower estimate bounds the L^2-norm of v in a small rectangle at the origin with a Sobolev norm of v in a sufficiently distant region.

The Carleman estimates we present here, stated in Theorem 4 below, are simple in the sense that they are established only in spaces L^pL^q with $p, q \in \{1, 2, \infty\}$ and are proved by means of elementary properties of the Fourier transform without appealing explicitly to the smoothing effects of the linearized equation or to more advanced tools in Harmonic Analysis like the L^2-boundedness of the maximal function associated to the inverse Fourier Transform [10,3] used in [5] and [2].

As far as the lower estimate is concerned, contrary to the case of the equation with positive dispersion, the sign of the nonlocal term in (1.1) does not favor a straightforward derivation of this estimate (Theorem 7, below). For this reason, for the equation with negative dispersion it is necessary to require a decay hypothesis stronger than that in [2]. However, we must point out that, as it was proved in [14], the fundamental solution of the Ostrovsky equation with negative dispersion decays as $e^{-cx^{3/2}}$ as $x \to +\infty$, and thus we are not able to assure the sharpness of our result.

The article is organized as follows: In Section 2 we prove that the exponential decay of the initial data is preserved in time, a necessary aspect in the application of the Carleman estimates, which are obtained in Section 3. In Section 4 we prove the lower estimate. Finally in Section 5 we prove Theorem 1.

We finish this introduction with some comments about the operator ∂_x^{-1} and the notations. If a function $f \in L^2_x \equiv L^2(\mathbb{R})$ is such that $\partial_x^{-1} f := \mathcal{F}^{-1}(\widehat{f}(\xi)/i\xi) \in L^2(\mathbb{R})$, then it is easy to see that $\int_{-\infty}^{\infty} f(x)\,dx = 0$ in the generalized sense (here $\widehat{\ }$ and $\mathcal{F}^{-1}$ denote respectively the Fourier transform and its inverse), and in this case $\partial_x^{-1} f$ has a continuous representative given by

$$\partial_x^{-1} f(x) = \int_{-\infty}^{x} f(x')\,dx' = -\int_{x}^{\infty} f(x')\,dx', \quad x \in \mathbb{R}. \tag{1.2}$$