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Asymptotic stability at infinity for bidimensional Hurwitz vector fields ☆

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ABSTRACT

Let $X : U \to \mathbb{R}^2$ be a differentiable vector field. Set $\mathrm{Spc}(X) = \{\text{eigenvalues of } DX(z)\colon z \in U\}$. This X is called Hurwitz if $\mathrm{Spc}(X) \subset \{z \in \mathbb{C}\colon \Re(z) < 0\}$. Suppose that X is Hurwitz and $U \subset \mathbb{R}^2$ is the complement of a compact set. Then by adding to X a constant v one obtains that the infinity is either an attractor or a repellor for $X + v$. That means: (i) there exists a unbounded sequence of closed curves, pairwise bounding an annulus the boundary of which is transversal to $X + v$, and (ii) there is a neighborhood of infinity with unbounded trajectories, free of singularities and periodic trajectories of $X + v$. This result is obtained after to proving the existence of $\tilde{X} : \mathbb{R}^2 \to \mathbb{R}^2$, a topological embedding such that $\tilde{X}$ equals X in the complement of some compact subset of U.

1. Introduction

A basic example of non-discrete dynamics on the Euclidean space is given by a linear vector field. This linear system is infinitesimally hyperbolic if every eigenvalue has nonzero real part, and it has well-known properties [16,3]. For instance, when the real part of its eigenvalues are negative (Hurwitz matrix), the origin is a global attractor rest point. In the nonlinear case, there has been a great interest in the local study of vector fields around their rest points [6,27,7,28]. However, in order to describe a global phase-portrait, as in [22,25,5,8,9] it is absolutely necessary to study its behavior in a neighborhood of infinity [19].

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The Asymptotic Stability at Infinity has been investigated with a strong influence of [18], where Olech showed a connection between stability and injectivity (see also [10,4,11,26,22]). This problem was also researched in [19,12,14,15,23,1]. In [12], Gutierrez and Teixeira consider C^1-vector fields $Y : \mathbb{R}^2 \to \mathbb{R}^2$, the linearizations of which satisfy (i) $\det(DY(z)) > 0$ and (ii) $\mathrm{Trace}(DY(z)) < 0$ in a neighborhood of infinity. By using [9], they prove that if Y has a rest point and the Index $\mathcal{I}(Y) = \int \mathrm{Trace}\ (DY) < 0$ (resp. $\mathcal{I}(Y) \geqslant 0$), then Y is topologically equivalent to $z \mapsto -z$ that is "the infinity is a repellor" (resp. to $z \mapsto z$ that is "the infinity is an attractor"). This Gutierrez–Teixeira's paper was used to obtain the next theorem, where $\mathrm{Spc}(Y) = \{\text{eigenvalues of } DY(z)\colon\ z \in \mathbb{R}^2 \setminus \overline{D}_\sigma\}$, and $\Re(z)$ is the real part of $z \in \mathbb{C}$.

Theorem 1 *(Gutierrez–Sarmiento). Let $Y : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1- map, where $\sigma > 0$ and $\overline{D}_\sigma = \{z \in \mathbb{R}^2\colon\ \|z\| \leqslant \sigma\}$. The following is satisfied:*

(i) *If for some $\varepsilon > 0$, $\mathrm{Spc}(Y) \cap (-\varepsilon, +\infty) = \emptyset$. Then there exists $s \geqslant \sigma$ such that the restriction $Y| : \mathbb{R}^2 \setminus \overline{D}_s \to \mathbb{R}^2$ is injective.*

(ii) *If for some $\varepsilon > 0$, the spectrum $\mathrm{Spc}(Y)$ is disjoint of the union $(-\varepsilon, 0] \cup \{z \in \mathbb{C}\colon\ \Re(z) \geqslant 0\}$. Then there exist $p_0 \in \mathbb{R}^2$ such that the point ∞ of the Riemann Sphere $\mathbb{R}^2 \cup \{\infty\}$ is either an attractor or a repellor of $z' = Y(z) + p_0$.*

Theorem 1, given in [14], has been extended to differentiable maps $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ in [13,15]. In both papers the eigenvalues also avoid a real open neighborhood of zero. In [23], the author examine the intrinsic relation between the asymptotic behavior of $\mathrm{Spc}(X)$ and the global injectivity of the local diffeomorphism given by X. He uses $Y_\theta = R_\theta \circ Y \circ R_{-\theta}$, where R_θ is the linear rotation of angle $\theta \in \mathbb{R}$, and (motivated by [11]) introduces the so-called B-condition [24,26], which claims:

for each $\theta \in \mathbb{R}$, there does not exist a sequence $(x_k, y_k) \in \mathbb{R}^2$ with $x_k \to +\infty$ such that $Y_\theta((x_k, y_k)) \to p \in \mathbb{R}^2$ and $DY_\theta(x_k, y_k)$ has a real eigenvalue λ_k satisfying $x_k\lambda_k \to 0$.

By using this, [23] improves the differentiable version of Theorem 1.

In the present paper we prove that the condition

$$\mathrm{Spc}(X) \subset \big\{z \in \mathbb{C}\colon\ \Re(z) < 0\big\}$$

is enough in order to obtain Theorem 1 for differentiable vector fields $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$.

Throughout this paper, $\mathbb{R}^2$ is embedded in the Riemann Sphere $\mathbb{R}^2 \cup \{\infty\}$. Thus $(\mathbb{R}^2 \setminus \overline{D}_\sigma) \cup \{\infty\}$ is the subspace of $\mathbb{R}^2 \cup \{\infty\}$ with the induced topology, and 'infinity' refers to the point ∞ of $\mathbb{R}^2 \cup \{\infty\}$. Moreover given $C \subset \mathbb{R}^2$, a closed (compact, no boundary) curve (1-manifold), $\overline{D}(C)$ (respectively $D(C)$) is the compact disc (respectively open disc) bounded by C. Thus, the boundaries $\partial \overline{D}(C)$ and $\partial D(C)$ are equal to C, homeomorphic to $\partial D_1 = \{z \in \mathbb{R}^2\colon\ \|z\| = 1\}$.

2. Statement of the results

For every $\sigma > 0$ let $\overline{D}_\sigma = \{z \in \mathbb{R}^2\colon\ \|z\| \leqslant \sigma\}$. Outside this compact disk we consider a differentiable vector field $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$. As usual, a *trajectory of X starting at $q \in \mathbb{R}^2 \setminus \overline{D}_\sigma$* is defined as the integral curve determined by a maximal solution of the Initial Value Problem $\dot{z} = X(z)$, $z(0) = q$. This is a curve $I_q \ni t \mapsto \gamma_q(t) = (x(t), y(t))$, satisfying:

- t varies on some open real interval containing the zero, the image of which $\gamma_q(0) = q$;
- $\gamma_q(t) \in \mathbb{R}^2 \setminus \overline{D}_\sigma$ and there exist the real derivatives $\frac{dx}{dt}(t)$, $\frac{dy}{dt}(t)$;
- $\dot{\gamma}_q(t) = (\frac{dx}{dt}(t), \frac{dy}{dt}(t))$, the velocity vector field of γ_q at $\gamma_q(t)$ equals $X(\gamma_q(t))$, and
- $I_q \subset \mathbb{R}$ is the maximal interval of definition.