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The effects of indefinite nonlinear boundary conditions on the structure of the positive solutions set of a logistic equation

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Abstract

We investigate a semilinear elliptic equation with a logistic nonlinearity and an indefinite nonlinear boundary condition, both depending on a parameter λ. Overall, we analyze the effect of the indefinite nonlinear boundary condition on the structure of the positive solutions set. Based on variational and bifurcation techniques, our main results establish the existence of three nontrivial non-negative solutions for some values of λ, as well as their asymptotic behavior. These results suggest that the positive solutions set contains an S-shaped component in some case, as well as a combination of a C-shaped and an S-shaped components in another case.

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Keywords: Semilinear elliptic problem; Indefinite weight; Nonlinear boundary condition; Variational methods; Bifurcation approach; Population dynamics

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1. Introduction and main results

Let Ω be a bounded and regular domain of $\mathbb{R}^N$ with $N \geq 2$. In this article, we consider the problem

$$\begin{cases} -\Delta u = \lambda\big(m(x)u - a(x)|u|^{p-2}u\big) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda b(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where:

- Δ is the usual Laplacian in $\mathbb{R}^N$
- $\lambda \in \mathbb{R}$
- $1 < q < 2 < p$ and if $N \geq 3$ then $p \leq 2^* = \frac{2N}{N-2}$
- $m, a \in L^\infty(\Omega)$
- $b \in L^\infty(\partial\Omega)$
- $\mathbf{n}$ is the outward unit normal to $\partial\Omega$.

Unless otherwise stated, we assume all over this article that $m^+ = \max\{m, 0\} \not\equiv 0$, $a \geq 0$, $a \not\equiv 0$ and $b \not\equiv 0$. A function $u \in H^1(\Omega)$ is called a *weak solution* (or simply a *solution*) of (P_λ) if it satisfies

$$\int_\Omega \nabla u \nabla \varphi - \lambda \int_\Omega m(x)u\varphi + \lambda \int_\Omega a(x)|u|^{p-2}u\varphi - \lambda \int_{\partial\Omega} b(x)|u|^{q-2}u\varphi = 0, \quad \forall \varphi \in H^1(\Omega).$$

By a standard bootstrap argument ([9, Theorem 9.11], [17, Theorem 2.2]), we can prove that a solution u of (P_λ) satisfies $u \in W^{2,r}_{\mathrm{loc}}(\Omega) \cap C^\theta(\overline{\Omega})$ with $r > N$ and $0 < \theta < 1$, so that by the weak maximum principle [9, Theorem 9.1], a nontrivial non-negative solution of (P_λ) is strictly positive in Ω. The main goal of this article is to study existence and multiplicity of nontrivial non-negative solutions of (P_λ) and their asymptotic behavior through a combination of variational and bifurcation arguments. Furthermore, we are interested in the shape of the positive solutions set associated to (P_λ).

Problem (P_λ) describes the steady state of solutions of the corresponding initial boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (d\nabla u) + \big(m(x) - a(x)|u|^{p-2}\big)u & \text{in } (0,\infty) \times \Omega, \\ u(0,x) = u_0(x) \geq 0 & \text{in } \Omega, \\ (d\nabla u) \cdot \mathbf{n} = b(x)|u|^{q-2}u & \text{on } (0,\infty) \times \partial\Omega. \end{cases} \tag{1.1}$$

From a biological viewpoint, (1.1) is related to population dynamics: the unknown function u denotes the population density of some species having an intrinsic growth (or decay) rate $m(x)$ and a logistic growth rate $m(x) - a(x)|u|^{p-2}$ either with self-limitation ($a(x) > 0$) or without limitation ($a(x) = 0$), see Cantrell and Cosner [5]. On the other hand, the boundary condition suggests that the flux rate $(d\nabla u) \cdot \mathbf{n}$ of the population on $\partial\Omega$ is incoming or outgoing, according to the sign of $b(x)$ and depends nonlinearly on $|u|^{q-2}u$. We note