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Finite time blow-up in nonlinear suspension bridge models

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Abstract

This paper settles a conjecture by Gazzola and Pavani [10] regarding solutions to the fourth order ODE $w^{(4)} + kw'' + f(w) = 0$ which arises in models of traveling waves in suspension bridges when $k > 0$. Under suitable assumptions on the nonlinearity f and initial data, we demonstrate blow-up in finite time. The case $k \leq 0$ was first investigated by Gazzola et al., and it is also handled here with a proof that requires less differentiability on f. Our approach is inspired by Gazzola et al. and exhibits the oscillatory mechanism underlying the finite-time blow-up. This blow-up is nonmonotone, with solutions oscillating to higher amplitudes over shrinking time intervals. In the context of bridge dynamics this phenomenon appears to be a consequence of mutually-amplifying interactions between vertical displacements and torsional oscillations.

MSC: 34A12; 35C07; 34C10

Keywords: Suspension bridge; Traveling waves; Fourth-order differential equation; Oscillations; Finite time blow-up

1. Introduction

The topic of suspension bridges is a celebrated area of applied mathematics filled with engineering marvels, as well as many dramatic events. One of the most notorious disasters is the Tacoma Narrows Bridge collapse of 1940. The collapse of the bridge had been previously explained by a resonance-like effect produced by a wind of under 80 km/h [1]; however, in recent

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literature (e.g. [10,13]) it has been demonstrated that resonance theory does not accurately describe these vibrational patterns. The phenomenon of self-amplifying oscillations in bridge dynamics is now recognized to be far more complex than originally believed; see the wonderful historical overview of existing theories in [7,13]. To explain these dynamics, several models have been proposed. The following model based on the Euler–Bernoulli beam equation was introduced by Lazer and McKenna in [14] and investigated further in [13]:

$$u_{tt} + u_{xxxx} + \gamma u^{+} = W(x,t), \quad x \in (0,L),\ t > 0. \tag{1}$$

In the above model u denotes vertical displacement, $L > 0$ is the length of the bridge, $u^{+} = \max\{u, 0\}$, γu^{+} represents the force from the cables treated as springs with a one-sided restoring force, and W accounts for additional forces such as weight and wind. In [15] McKenna and Walter investigated traveling waves for this model. After some normalization, traveling wave solutions to (1) necessarily satisfy

$$w''''(t) + kw''(t) + f\big(w(t)\big) = 0 \tag{2}$$

with $f(s) = (s+1)^{+} - 1$. In [4], a smooth analog of this nonlinearity given by $f(t) = e^{t} - 1$ was considered.

Recent research (e.g. [7]) shows that the stability of a bridge can be critically affected by torsional oscillations, in particular, their interactions with vertical displacements. Consequently, a one-dimensional model in the above interpretation does not accurately describe large oscillations, as twisting effects would not be taken into account. Indeed, traveling wave solutions corresponding to (2) are global when $f(s) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, $f(s)s > 0$ for $s \neq 0$, and $f(s)$ has at most linear growth either as $s \to +\infty$ or as $s \to -\infty$, as it was shown in [3].

Observations of actual bridge oscillations (Millennium Bridge [2]) and collapses (Tacoma Narrows Bridge [6]) reinforce the idea that torsional and vertical oscillations in suspension bridges are coupled. To model this interaction, Drábek et al. [5] introduced a second unknown function to measure potentially unbounded torsional effects. Subsequently, it was suggested in [9] that the coupling mechanism be incorporated into a one-dimensional model by allowing the forcing term f to take arbitrarily large negative values. In this new model *positive values of w correspond to vertical oscillations, while negative ones describe torsional deformations*. A suitable function f would necessarily be sign preserving, e.g. $f(s) = s^3 + s$. For an extensive overview of the theory for ODEs of the form (2) see the book [16] by Peletier and Troy.

The study of traveling waves for (1) when vertical and torsional oscillations are unbounded has been a challenging open problem. The current paper investigates finite time blow-up of solutions to Eq. (2) when f is a locally Lipschitz function unbounded as $|t| \to \infty$.

In their paper [10] Gazzola and Pavani offered an innovative proof of blow-up for the case $k \le 0$; this range for k corresponds to models of beams in tension where $-k \ge 0$ represents the tension [11]. The scenario $k > 0$ corresponds to traveling wave solutions

$$u(t,x) = w(x + ct)$$

of the Euler–Bernoulli equation, with $k = c^2$ where c is the speed of the wave. Numerical evidence strongly supports the blow-up for $k > 0$, as presented in [10]. This case, however, remained open until now since positive values of the parameter k critically alter some intrinsic invariants associated with the ODE. We settle the blow-up conjecture when