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# Regularity of the extremal solutions associated to elliptic systems [☆]

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**Abstract**

We examine the two elliptic systems given by

$$(G)_{\lambda,\gamma} \qquad -\Delta u=\lambda f'(u)g(v), \qquad -\Delta v=\gamma f(u)g'(v) \quad \text{in } \Omega,$$

and

$$(H)_{\lambda,\gamma} \qquad -\Delta u=\lambda f(u)g'(v), \qquad -\Delta v=\gamma f'(u)g(v) \quad \text{in } \Omega,$$

with zero Dirichlet boundary conditions and where $\lambda$, $\gamma$ are positive parameters. We show that for general nonlinearities $f$ and $g$ the extremal solutions associated with $(G)_{\lambda,\gamma}$ are bounded, provided $\Omega$ is a convex domain in $\mathbb{R}^N$ where $N \le 3$. In the case of a radial domain, we show the extremal solutions are bounded provided $N < 10$. The extremal solutions associated with $(H)_{\lambda,\gamma}$ are bounded in the case where $f$ is a general nonlinearity, $g(v)=(v+1)^q$ for $1<q<\infty$ and when $\Omega$ is a bounded convex domain in $\mathbb{R}^N$ for $N \le 3$. Certain regularity results are also obtained in higher dimensions for $(G)_{\lambda,\gamma}$ and $(H)_{\lambda,\gamma}$ for the case of explicit nonlinearities of the form $f(u)=(u+1)^p$ and $g(v)=(v+1)^q$.


*Keywords:* Elliptic systems; Extremal solutions; Stable solutions; Regularity of solutions; Radial solutions

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## 1. Introduction

We examine the following systems:

$$(G)_{\lambda,\gamma} \qquad \begin{cases} -\Delta u = \lambda f'(u) g(v) & \Omega \\ -\Delta v = \gamma f(u) g'(v) & \Omega, \\ u = v = 0 & \partial\Omega \end{cases}$$

and

$$(H)_{\lambda,\gamma} \qquad \begin{cases} -\Delta u = \lambda f(u) g'(v) & \Omega \\ -\Delta v = \gamma f'(u) g(v) & \Omega, \\ u = v = 0 & \partial\Omega \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ and $\lambda, \gamma > 0$ are positive parameters. The nonlinearities $f$ and $g$ will satisfy various properties but will always at least satisfy

$(R)$ $\quad f$ is smooth, increasing and convex with $f(0) = 1$ and $f$ superlinear at $\infty$.

We begin by recalling the scalar analog of the above systems. Given a nonlinearity $f$ which satisfies (R), the following equation

$$(Q)_\lambda \qquad \begin{cases} -\Delta u = \lambda f(u) & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

is now quite well understood whenever $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$. See, for instance, [1–5,8,10,12,15]. We now list the properties one comes to expect when studying $(Q)_\lambda$. It is well known that there exists a critical parameter $\lambda^* \in (0, \infty)$, called the extremal parameter, such that for all $0 < \lambda < \lambda^*$ there exists a smooth, minimal solution $u_\lambda$ of $(Q)_\lambda$. Here the minimal solution means in the pointwise sense. In addition for each $x \in \Omega$ the map $\lambda \mapsto u_\lambda(x)$ is increasing in $(0, \lambda^*)$. This allows one to define the pointwise limit $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ which can be shown to be a weak solution, in a suitably defined sense, of $(Q)_{\lambda^*}$. For this reason $u^*$ is called the extremal solution. It is also known that for $\lambda > \lambda^*$ there are no weak solutions of $(Q)_\lambda$. Also one can show the minimal solution $u_\lambda$ is a semi-stable solution of $(Q)_\lambda$ in the sense that

$$\int_\Omega \lambda f'(u_\lambda)\psi^2 \le \int_\Omega |\nabla\psi|^2, \quad \forall \psi \in H_0^1(\Omega).$$

A question that has attracted a lot of attention is the regularity of the extremal solution. It is known that the extremal solution can be a classical solution or it can be a singular weak solution. We now list some results in this direction:

- ([12]) $u^*$ is bounded if $f$ satisfies (R) and $N \le 3$.
- ([3]) $u^*$ is bounded if $f$ satisfies (R) (can drop the convexity assumption) and $\Omega$ a convex domain in $\mathbb{R}^4$.
- ([4]) $u^*$ is bounded if $\Omega$ is a radial domain in $\mathbb{R}^N$ with $N < 10$ and $f$ satisfies (R) (can drop the convexity assumption).