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Clustering and asymptotic behavior in opinion formation ☆

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Abstract

We investigate the long time behavior of models of opinion formation. We consider the case of compactly supported interactions between agents which are also non-symmetric, including for instance the so-called Krause model. Because of the finite range of interaction, convergence to a unique consensus is not expected in general. We are nevertheless able to prove the convergence to a final equilibrium state composed of possibly several local consensus. This result had so far only been conjectured through numerical evidence. Because of the non-symmetry in the model, the analysis is delicate and is performed in two steps: First using entropy estimates to prove the formation of stable clusters and then studying the evolution in each cluster. We study both discrete and continuous in time models and give rates of convergence when those are available.

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1. Introduction

Thanks to the development of social media, the dynamics of opinion formation has recently generated much interest [1–7]. Through a complex network of interactions emerge groups with various opinions. Several questions arise from such dynamics, for instance how groups are formed and how many of them will survive throughout time.

Different models have been introduced to study opinion dynamics [2,5,8–11,1,12,13]. In this paper, we focus on a widely-used model referred to as the *consensus model* [8,14–17]. The consensus model describes the evolution of N agents which tend to have a similar *opinion* as the one of their close neighbors. Each opinion is represented by a quantity $x_i \in \mathbb{R}^d$ (a scalar or a vector) and evolves according to the following dynamics:

$$\dot{x}_i = \frac{\sum_j \phi_{ij}(x_j - x_i)}{\sum_j \phi_{ij}}, \quad \phi_{ij} = \phi\big(|x_j - x_i|^2\big). \tag{1.1}$$

Here, ϕ is the so-called influence function, it is a non-negative function strictly positive at the origin (i.e. $\phi(0) > 0$). The greater ϕ_{ij} is, the more the agent i is influenced by j (and vice-versa). Without loss of generality, one can assume that $\phi(0) = 1$.

Several studies have been conducted to study numerically the long time behavior of the consensus model [15,18,19]. It has been observed that the dynamics generate concentration of opinions (also called *clusters*) and that, after a transient period, the configuration stabilizes. The goal of this manuscript is to prove analytically those observations. One of the main difficulties in studying the consensus model is the lack of conserved quantities. For instance, the *total momentum*, $\frac{1}{N}\sum_{i=1}^N x_i$, is not preserved in time. This lack of conservation is due to the non-symmetry in the interaction. Writing the model as

$$\dot{x}_i = a_{ij}(x_j - x_i), \quad a_{ij} = \frac{\phi_{ij}}{\sum_j \phi_{ij}}, \tag{1.2}$$

we find that $a_{ij} \neq a_{ji}$ in general. Thus, the influence between agents is not symmetric.

Depending on the interaction function ϕ, one can have very different dynamics. When the interaction function ϕ is globally supported (i.e. $\phi(r) > 0$ for all $r \geq 0$) meaning that every agent interacts with each other, the dynamics is easily understood: the system converges to a so-called *consensus* [19]. In other words, there exists one opinion x_∞ such that

$$x_i(t) \xrightarrow{t\to\infty} x_\infty, \quad \text{for all } i. \tag{1.3}$$

However the case with a compactly supported interaction function ϕ is more relevant and realistic from a modeling perspective. Agents then only interact with those having relatively similar opinions and this makes the corresponding analysis more delicate. One immediately sees that the previous simple scenario cannot always take place: Just take two agents with initial positions $x_1(0)$ and $x_2(0)$ s.t. $\phi(|x_1 - x_2|) = 0$. However in that elementary case, we still have formation of what one would call *local consensus*. This means that the opinions $x_i(t)$ still converge in time to some limit $\bar{x}_i$. All the limits $\bar{x}_i$ are not necessarily equal though.

The question we wish to investigate in this paper is whether we always have local consensus or if some other, more complicated, asymptotic behaviors can manifest. It turns out that for the interaction kernels ϕ used in practice, we can prove that local consensus always takes place.