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Fredholm operators, evolution semigroups, and periodic solutions of nonlinear periodic systems ☆

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Abstract

Let X be a Banach space and L the generator of the evolution semigroup associated with the τ-periodic evolutionary process $\{U(t,s)\}_{t\geq s}$ on the space $P_\tau(X)$ of all τ-periodic continuous X-valued functions. We give criteria for the existence of periodic solutions to nonlinear systems of the form $Lp = -\epsilon F(p,\epsilon)$ under the condition that 1 is a normal eigenvalue of the monodromy operator $U(\tau,0)$. The proof is based on a new decomposition of the space $P_\tau(X)$ by constructing a right inverse of L.

MSC: primary 34G20, 46N20, 35B10; secondary 47A10, 47A53, 47D06, 35K58

Keywords: Nonlinear periodic system; Evolution semigroup; Fredholm operator; Periodic solution; Normal eigenvalue; Decomposition

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1. Introduction

Let X be a complex Banach space and let $\{U(t,s)\}_{t\geq s}$ be a τ-periodic evolutionary process on X. Denote by L the generator of the evolution semigroup $\{T^h\}_{h\geq 0}$ associated with $\{U(t,s)\}_{t\geq s}$ on the space $P_\tau(X)$ of all τ-periodic continuous X-valued functions. Define the monodromy operator (or sometimes, period map, Poincaré map) $V(t)$ of the evolutionary process by $V(t)=U(t,t-\tau)$.

We consider the existence of periodic solutions to nonlinear systems of the form

$$Lp=-\epsilon F(p,\epsilon), \quad p\in\mathcal{D}(L), \tag{1}$$

where F is a nonlinear operator, defined on an open subset $\Omega\subset P_\tau(X)\times\mathbb{R}$, with certain conditions (see Section 6). The main purpose of this paper is to present a general theory for the existence of periodic solutions to Eq. (1) under the condition that 1 is a normal eigenvalue of $V(0)$.

Eq. (1) is a generalization of the nonlinear τ-periodic systems of the form

$$\frac{dx}{dt}=A(t)x+\epsilon f(t,x,\epsilon). \tag{2}$$

In finite-dimensional cases, we have no problem with the existence of evolutionary process for this equation. However, it is very difficult to prove the existence of evolutionary process for $x'=A(t)x$ in infinite-dimensional cases. As in the book [2] we will not pursue the problem of existence of evolutionary process, but deal with mild solutions under the starting assumption that the evolutionary process exists. Here we only show a simple example that if $A(t)=A+\alpha(t)I$, where A is the generator of a C_0-semigroup on X, and $\alpha(t)$ is a τ-periodic continuous scalar-valued function, then $U(t,s)=e^{\int_s^t\alpha(r)dr}e^{(t-s)A}$. The reader may refer to [2,17–19] for more general cases.

There are many references on the existence of periodic solutions of semilinear time periodic evolution equations in infinite-dimensional cases; for example, refer to [7,10–13,15,16] and their references. The book [13] deals with criteria on the existence of periodic or almost periodic solutions via the evolution semigroups for linear systems. In the paper [15] the existence of periodic solutions is discussed by using fixed point theorems for nonlinear (linear) ordinary, functional or partial differential equations and in the references therein periodic solution theory of nonlinear evolution equations bas been extensively studied via various methods. Our paper is mainly related to Henry's book [12], where for a certain class of parabolic problems criteria on the existence of periodic solutions are presented in Theorems 8.3.1 and 8.3.2 of [12] under the condition that $1\in\rho(V(0))$ or 1 is an isolated simple eigenvalue of $V(0)$. Lunardi's book [16, Proposition 6.3.12] gives formulae of periodic solutions for inhomogeneous linear periodic system under the condition that 1 is a simple pole of the resolvent of $V(0)$.

On the other hand, the existence of periodic solutions to Eq. (2) in finite-dimensional cases has been investigated via various methods; refer to [5,7–9,20,24] and the references therein. No restriction on the singularity of $I-V(0)$ is assumed for the nonlinear perturbation of periodic solutions to the homogeneous linear periodic equation in [5, Theorem 3.1 in Chapter 14], which is introduced as [7, Theorem 6.1.4]. The similar results are contained in [8, Part II].

To obtain criteria for the existence of periodic solutions to Eq. (1), we will combine the alternative method in [3,4] and [9] with the implicit function theorem. To apply the alternative