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Integral representation and asymptotic behavior of harmonic functions in half space

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Abstract

Using Carleman's formula of a harmonic function in the half space and Nevanlinna's representation of a harmonic function in the half sphere, we prove that a harmonic function, whose positive part satisfies a slowly growing condition, can be represented by a certain integral. This improves some classical Poisson integrals for harmonic functions.

Keywords: Carleman's formula; Nevanlinna's representation; Integral representation; Growth

1. Introduction

If D is a finitely connected plane domain, the holomorphic function with sufficiently well-behaved approaching the boundary ∂D can be recovered from its boundary values by the Cauchy integral formula [1]. The corresponding results on the disc were obtained by Riesz and Riesez [2], while those on other domains were derived by Smirnov [3]. Their results state that this class of functions coincides with the Hardy class [4]. When the boundary values are only known on a

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proper subset I of ∂D, having positive measure, it becomes a special case of a classical ill-posed issue namely the Cauchy problem of the Laplace equation. In 1926, Carleman [5] solved this problem on a plane domain D of special form, reconstructing the analytic function on a domain by its values on a subset of the boundary.

Integral representations of holomorphic functions play an essential role in the classical theory of functions of one complex variable and multidimensional complex variables. Carleman's formulas, unlike the Cauchy formula, restore a function holomorphic in a domain D by its value on a part of the boundary ∂D. A convenient reference in [6], Carleman's formula establishes a connection between the modulus and the zeros of analytic function in the half plane [7]. Its relation to Poisson's integral formula was brought out by brothers Nevanlinna in a classical paper [8], and further analyzed by Nevanlinna in [9], a paper that deserves to be better known. Aizenberg et al. [4] describe the class of holomorphic functions, with the sufficient condition, which can be represented by Carleman formula. A Carleman type formula of meromorphic functions, with remainder term, defined on rectangle was obtained explicitly in [10]. Its modification to the holomorphic functions, which are real on a side of a rectangle, was proved and applied to the study of the Riemann-ζ function [11].

Zhang et al. [12] generalize the Carleman formula of harmonic function from the half plane to the half space and Nevanlinna's representation from the half disk to the half sphere. It motivates us to study the recovering results of harmonic function from partial boundary values in higher dimensions. We assume the positive part of the harmonic function under study satisfies some conditions.

The article is organized as follows. Section 2 gives a brief introduction and terminology. The main results are devoted to Theorem 3.1 and Theorem 3.2 in Section 3. The proofs of these theorems are studied in Section 4.

2. Preliminaries

First of all, we introduce some notations and terminology. Let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space. For $x \in \mathbb{R}^n$ the Euclidean norm is defined by $|x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Let $\mathbb{H} := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$ be the upper half space, where $x' \in \mathbb{R}^{n-1}$ stand for the projection of x onto the hyperplane $\partial\mathbb{H} := \mathbb{R}^{n-1}$. A function $u \in C^2(\mathbb{H})$ is called a harmonic function if u satisfies the Laplace equation $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0$. We shall concern those harmonic functions whose positive part does not have a great growth rate. In the following, denote $u(y') := u(y', 0)$, $y' \in \mathbb{R}^{n-1}$.

Let $\mathbb{H}(2)$ be the harmonic functions $u : \mathbb{R}^n \to \mathbb{R}$ that satisfy

$$\int_{\mathbb{H}} \frac{x_n u^+(x)}{1 + |x|^n} dx < \infty, \tag{1}$$

where $u^+(x) = \max\{u(x), 0\}$ is the positive part of $u(x)$.

Recall that the Poisson kernel on the upper half-space $\mathbb{H}$ is given by

$$P_{\mathbb{H}}(x, y') := \frac{2x_n}{n\omega_n} \frac{1}{|x - y'|^n},$$