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Almost global existence for nonlinear wave equations in an exterior domain in two space dimensions

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Dedicated to Professor Yoshihiro Shibata on the occasion of his 60th birthday

Abstract

In this paper we deal with the exterior problem for a system of nonlinear wave equations in two space dimensions, assuming that the initial data is small and smooth. We establish the same type of lower bound of the lifespan for the problem as that for the Cauchy problem, despite of the weak decay property of the solution in two space dimensions.

MSC: 35L70; 35L20

Keywords: Exterior problem; Nonlinear wave equation; Lifespan

1. Introduction and statement of main results

Let Ω be an unbounded domain in $\mathbf{R}^n$ $(n \geq 2)$ with compact and smooth boundary $\partial\Omega$. We put $\mathcal{O} := \mathbf{R}^n \setminus \Omega$, which is called an obstacle and is assumed to be non-empty. We consider the mixed problem for a system of nonlinear wave equations:

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$$\big(\partial_t^2 - \Delta\big)u_i = F_i(\partial u, \nabla_x \partial u), \quad (t,x) \in (0,\infty) \times \Omega, \tag{1.1}$$

$$u(t,x) = 0, \quad (t,x) \in (0,\infty) \times \partial\Omega, \tag{1.2}$$

$$u(0,x) = \varepsilon\phi(x), \quad \partial_t u(0,x) = \varepsilon\psi(x), \quad x \in \Omega \tag{1.3}$$

for $i = 1, \dots, N$, where $u = (u_1, u_2, \dots, u_N)$ is an unknown function, $\Delta = \sum_{j=1}^n \partial_j^2$, $\partial_t = \partial_0 = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j = 1, \dots, n)$, and $\varepsilon > 0$. We assume $\phi, \psi \in C_0^\infty(\overline{\Omega}; \mathbf{R}^N)$, namely, they are smooth functions on $\overline{\Omega}$ vanishing outside some ball. We also assume that (1.1) is quasi-linear, namely each F_i has the form

$$F_i(\partial u, \nabla_x \partial u) = \sum_{j=1}^N \sum_{(a,b)\neq(0,0)} c_{ij}^{ab}(\partial u)\partial_a \partial_b u_j + \widetilde{F}_i(\partial u),$$

where c_{ij}^{ab} and $\widetilde{F}_i$ are smooth functions satisfying

$$c_{ij}^{ab}(\partial u) = O\big(|\partial u|^{q-1}\big), \qquad \widetilde{F}_i(\partial u) = O\big(|\partial u|^q\big), \quad 1 \le i, j \le N, \ 0 \le a, b \le 2, \tag{1.4}$$

around $\partial u = 0$ for some integer $q \ge 2$. In the following we always assume the energy symmetric condition, i.e.

$$c_{ij}^{ab}(\partial u) = c_{ji}^{ab}(\partial u) \quad \text{and} \quad c_{ij}^{ab}(\partial u) = c_{ij}^{ba}(\partial u)$$

hold for $1 \le i, j \le N$ and $0 \le a, b \le 2$ $((a,b) \neq (0,0))$, so that the hyperbolicity of the system is assured. We suppose, in addition, that (ϕ, ψ, F) satisfies the compatibility condition to infinite order for the mixed problem (1.1)–(1.3), that is, $(\partial_t^j u)(0,x)$, formally determined by (1.1) and (1.3), vanishes on $\partial\Omega$ for any non-negative integer j (notice that the values $(\partial_t^j u)(0,x)$ are determined by (ϕ, ψ, F) successively; for example we have $\partial_t^2 u(0) = \varepsilon\Delta_x \phi_i + F_i(\varepsilon(\psi, \nabla_x \phi), \varepsilon\nabla_x(\psi, \nabla_x \phi))$, and so on).

It was firstly shown by Shibata and Tsutsumi [21] that the mixed problem for (1.1)–(1.3) admits a unique global solution for sufficiently small initial data, when either $n \ge 6$ and $q \ge 2$ or $3 \le n \le 5$ and $q \ge 3$, provided $\mathcal{O}$ is non-trapping. Although the dispersive property becomes weaker as the spatial dimension is lower, there are already many contributions for the case where $3 \le n \le 5$ and $q = 2$ (see [4–6,9–11,15,17–19] and the references cited therein).

However, up to the author's knowledge, there are only a few results for the case $n = 2$; in [22] the global solvability is considered when the nonlinear term depends only on u itself; in [24] the blow-up of solutions is studied, and when $F = |\partial_t u|^3$ in (1.1), the lifespan T_ε is estimated as $T_\varepsilon \le \exp(A\varepsilon^{-2})$ with some positive constant A. Here, T_ε is defined by the supremum of all $T > 0$ such that a classical solution to the problem (1.1)–(1.3) exists in $[0,T) \times \overline{\Omega}$. The latter result suggests us that the cubic nonlinearity is on the critical level concerning the global existence theorem for small initial data when $n = 2$.

Therefore, it is natural to ask whether the above upper bound of the lifespan is optimal with respect to ε or not. The aim of this paper is to establish an affirmative answer to this question as follows.

Theorem 1.1. *Let $n = 2$ and let $\phi, \psi \in C^\infty(\overline{\Omega}; \mathbf{R}^N)$ vanish outside certain ball. Assume that (ϕ, ψ, F) satisfies the compatibility condition to infinite order for the problem (1.1)–(1.3), $\mathcal{O}$ is*