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Čech cohomology of attractors of discrete dynamical systems ☆

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Abstract

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism and K an asymptotically stable attractor for f. The aim of this paper is to study when the inclusion of K in its basin of attraction $\mathcal{A}(K)$ induces isomorphisms in Čech cohomology. We show that (i) this is true if coefficients are taken in $\mathbb{Q}$ or $\mathbb{Z}_p$ (p prime) and (ii) it is true for integral cohomology if and only if the Čech cohomology of K or $\mathcal{A}(K)$ is finitely generated. We compute the Čech cohomology of periodic point free attractors of volume-contracting $\mathbb{R}^3$-homeomorphisms and present applications to quite general models in population dynamics.

MSC: 54H20; 55N05; 55P55

Keywords: Čech cohomology; Attractor; Basin of attraction; Homeomorphism

1. Introduction

Let K be a compact attractor of a flow and $\mathcal{A}(K)$ its basin of attraction. There are many papers in the literature relating the homotopy properties of $\mathcal{A}(K)$ and K. Since K may have a very complicated topological structure, the homotopy theory that best suits the study of this problem is shape theory, which can be thought of as a sort of Čech homotopy theory (see [2], [1], [3] or [15]). If the flow is defined in a nice space (a manifold or more generally an ANR) the main conclusions are that the inclusion i of K in $\mathcal{A}(K)$ is a shape equivalence and that K has the shape of a finite polyhedron (see for instance [7], [4], [6] or [12]). In particular, i induces

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isomorphisms in Čech cohomology. The proofs of these facts depend in an essential way on the homotopies that a flow provides for free.

In the case of discrete dynamical systems few results are known about the homotopical relationship between K and $\mathcal{A}(K)$, and — due to the absence of the homotopies which a flow would naturally provide — they require strong conditions on the homeomorphism or on the attractor (see [5,16,19]). Such conditions are not useful in practice because *a priori* it is not known how strange the attractor can be, although in low dimensions the situation is slightly more tractable [19].

Given the situation just described it is profitable to be less ambitious and concentrate on the relation between the Čech cohomology of K and $\mathcal{A}(K)$. One of the difficulties that arises is related to the fact that, unless the attractor has some kind of movability property (which is, again, difficult to check), information may be lost when the whole inverse sequence used to compute $\check{H}^*(K)$ is replaced by its inverse limit. In this paper we show that when coefficients are taken in $\mathbb{Q}$ or $\mathbb{Z}_p$ (p prime) this issue disappears and the inclusion of K in $\mathcal{A}(K)$ induces isomorphisms in Čech cohomology (Theorem 1); we also characterize when the same holds true for integral cohomology (Theorem 2):

Theorem 1. *Let $K \subseteq \mathbb{R}^n$ be an attractor for a homeomorphism f. Then the inclusion $i : K \longrightarrow \mathcal{A}(K)$ induces isomorphisms $i^* : H^d(\mathcal{A}(K); \mathbb{Q}) \longrightarrow \check{H}^d(K; \mathbb{Q})$. Moreover, both $\check{H}^d(K; \mathbb{Q})$ and $H^d(\mathcal{A}(K); \mathbb{Q})$ are finite dimensional vector spaces.*

The same holds true when coefficients are taken in $\mathbb{Z}_p$ with p prime.

Theorem 2. *Let $K \subseteq \mathbb{R}^n$ be an attractor for a homeomorphism f. The following are equivalent:*

(1) *the inclusion $i : K \subseteq \mathcal{A}(K)$ induces isomorphisms in Čech cohomology with $\mathbb{Z}$ coefficients,*
(2) *K has finitely generated Čech cohomology with $\mathbb{Z}$ coefficients,*
(3) *$\mathcal{A}(K)$ has finitely generated cohomology with $\mathbb{Z}$ coefficients.*

As an application we consider volume contracting homeomorphisms of $\mathbb{R}^3$ and compute the cohomology of attractors having no fixed or periodic points (Theorem 17). The reader may find in [9] a complete exposition of the fixed point index and Lefschetz theory. We then use this to study attractors of some periodic equations in $\mathbb{R}^3$ (Theorem 18), in particular quite general 3-dimensional models of population dynamics.

Background definitions and notation. Unless otherwise stated, f will always denote a homeomorphism of $\mathbb{R}^n$. An *attractor* for f is a compact set K with the following properties:

(1) $f(K) = K$ (K is *invariant*),
(2) K has a neighborhood U such that for every compact set $P \subseteq U$ and every neighborhood V of K there exists k_0 with the property that $f^k(P) \subseteq V$ for every $k \geq k_0$ (K *attracts* compact subsets of U).

The maximal U such that (2) holds is called the *basin of attraction* of K and denoted $\mathcal{A}(K)$. It is always an invariant, open subset of $\mathbb{R}^n$. The usual way of proving that f has an attractor is by finding a compact set $N \subseteq \mathbb{R}^n$ such that $f(N) \subseteq \operatorname{int} N$, for then it can be shown that f has an