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Asymptotic expansions for elliptic-like regularizations of semilinear evolution equations

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Abstract

Consider in a real Hilbert space H the Cauchy problem (P_0): $u'(t) + Au(t) + Bu(t) = f(t)$, $0 \leq t \leq T$; $u(0) = u_0$, where $-A$ is the infinitesimal generator of a C_0-semigroup of contractions, B is a nonlinear monotone operator, and f is a given H-valued function. Inspired by the excellent book on singular perturbations by J.L. Lions, we associate with problem (P_0) the following regularization (P_ε): $-\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t)$, $0 \leq t \leq T$; $u(0) = u_0$, $u'(T) = u_T$, where $\varepsilon > 0$ is a small parameter. We investigate existence, uniqueness and higher regularity for problem (P_ε). Then we establish asymptotic expansions of order zero, and of order one, for the solution of (P_ε). Problem (P_ε) turns out to be regularly perturbed of order zero, and singularly perturbed of order one, with respect to the norm of $C([0, T]; H)$. However, the boundary layer of order one is not visible through the norm of $L^2(0, T; H)$.

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Keywords: Semilinear evolution equation; Elliptic-like regularization; Higher regularity of solutions; Singular perturbation; Asymptotic expansion

1. Introduction

We will always assume that all operators are single-valued. Let H be a real Hilbert space with scalar product $(\cdot, \cdot)$ and the induced norm $\|\cdot\|$. Denote by (P_0) the following Cauchy problem

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$$\begin{cases} u'(t) + Au(t) + Bu(t) = f(t), & 0 \le t \le T, \qquad (E) \\ u(0) = u_0, \end{cases} \tag{P_0}$$

where $T > 0$ is a given time instant, $u_0 \in H$ is a given initial state, $f: [0, T] \to H$, $A: D(A) \subset H \to H$ is a linear operator, such that

(H1) A is a maximal monotone linear operator, or equivalently, $-A$ is the infinitesimal generator of a C_0-semigroup of contractions on H, say $\{S(t): H \to H; t \ge 0\}$;
(H2) $B: H \to H$ is a Lipschitz monotone operator, with Lipschitz constant C.

So equation (E) is a semilinear one.

A typical example of problem (P_0) is the semilinear heat equation when $-A$ is the Laplace operator Δ with some homogeneous boundary conditions.

Let $Q: D(Q) \subset H \to H$ be a maximal monotone operator. For every $\lambda > 0$, set

$$J_\lambda = (I + \lambda Q)^{-1} \quad \text{and} \quad Q_\lambda = \frac{1}{\lambda}(I - J_\lambda),$$

where J_λ is called the *resolvent* of Q, and Q_λ is the *Yosida approximation* of Q.

We recall that if $Q: D(Q) \subset H \to H$ is a maximal monotone operator, then, for all $\lambda > 0$, the range of $I + \lambda Q$ is the whole H, $I + \lambda Q$ is invertible and its inverse is Lipschitz with the Lipschitz constant equal to 1.

For information on the theory of monotone operators and semigroups of operators, see, e.g., [3,4,6,8–10].

Inspired by J.L. Lions [7], we consider the second order equation

$$\begin{cases} -\varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t), & 0 \le t \le T, \\ u(0) = u_0, & u'(T) = u_T. \end{cases} \tag{P_ε}$$

We are interested to know if the solution u_ε of problem (P_ε) approximates the solution u of problem (P_0). We will see that this is indeed the case, i.e., $\|u_\varepsilon - u\|_{C([0,T];H)}$ tends to zero as $\varepsilon \to 0^+$. However, when we establish the asymptotic expansion of order one for the solution of problem (P_ε), a boundary layer of order one occurs with respect to the norm of $C([0, T]; H)$, however, this boundary layer is not visible with respect to the norm of $L^2(0, T; H)$.

Lions explained (see [7, p. IX]) that sometimes it might be useful to consider regularizations of problem (P_0) that provide good solutions approximating the solution of (P_0) for ε small. This regularization method was also called by Lions the method of artificial viscosity (due to the additional term involving ε).

It is natural to call problem (P_ε) an elliptic-like regularization of problem (P_0), since (P_ε) is an elliptic type equation in a particular case when $-A$ is the Laplace operator Δ with the homogeneous Dirichlet boundary condition (see, e.g., [2, pp. 209–221] for the elliptic regularizations in this particular case). We mention that elliptic-like regularization (P_ε) is essentially different from the elliptic-like regularization discussed by the authors in [1].

2. Preliminaries

Consider the following Cauchy problem

$$\begin{cases} u'(t) + Qu(t) = f(t), & 0 < t < T, \qquad (F) \\ u(0) = u_0, \end{cases} \tag{P}$$