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Well-posedness for the Navier-slip thin-film equation in the case of complete wetting

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Abstract

We are interested in the thin-film equation with zero-contact angle and quadratic mobility, modeling the spreading of a thin liquid film, driven by capillarity and limited by viscosity in conjunction with a Navier-slip condition at the substrate. This degenerate fourth-order parabolic equation has the contact line as a free boundary. From the analysis of the self-similar source-type solution, one expects that the solution is smooth only as a function of *two* variables (x, x^β) (where x denotes the distance from the contact line) with $\beta = \frac{\sqrt{13}-1}{4} \approx 0.6514$ irrational. Therefore, the well-posedness theory is more subtle than in case of *linear* mobility (coming from Darcy dynamics) or in case of the *second*-order counterpart (the porous medium equation).

Here, we prove global existence and uniqueness for one-dimensional initial data that are close to traveling waves. The main ingredients are maximal regularity estimates in weighted L^2-spaces for the linearized evolution, after suitable subtraction of $a(t) + b(t)x^\beta$-terms.

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1. Introduction

1.1. The free boundary problem to the thin-film equation

We study regularity and stability properties of the free boundary problem

$$\partial_t h + \partial_z\big(h^2\partial_z^3 h\big) = 0 \quad \text{for } t > 0,\ z \in \big(Z_0(t), \infty\big), \tag{1.1a}$$

$$h = \partial_z h = 0 \quad \text{for } t > 0,\ z = Z_0(t), \tag{1.1b}$$

$$\dot{Z}_0(t) = \lim_{z \searrow Z_0(t)} h\partial_z^3 h \quad \text{for } t > 0. \tag{1.1c}$$

Eq. (1.1a) is a thin-film equation that can be derived in a lubrication approximation from the underlying Navier–Stokes equations of a two-dimensional viscous thin film on a one-dimensional flat solid [1,2]. The function $h(t,z)$ describes the height of the film, whereas $Z_0(t)$ denotes the position of the free boundary, which, in two space dimensions, corresponds to the contact line separating the three phases gas, liquid, and solid. Then $h = 0$ for $z = Z_0(t)$ merely defines the position of the contact line, while $\partial_z h = 0$ for $z = Z_0(t)$ enforces a zero contact angle between the liquid-gas and liquid–solid interfaces. In a quasi-static model, the contact angle is determined by a local equilibrium of surface tensions at the interfaces solid-gas, solid-liquid, and liquid-gas. The assumption $\partial_z h = 0$ at $z = Z_0(t)$ then implies that a global equilibrium is never attained, which is why the film will not stop spreading. This situation is commonly referred to as 'complete wetting regime' as opposed to the case of partial wetting, i.e. $\partial_z h \neq 0$.

Eq. (1.1a) has the form of a continuity equation

$$\partial_t h + \partial_z(hV) = 0,$$

with $V = h\partial_z^3 h$ the velocity with which the film height h is transported. In fact, within the lubrication approximation, V is the vertically averaged horizontal velocity of the fluid. The third boundary condition (1.1c) then states that the boundary value V_0 of the velocity V at the contact line equals the velocity $\dot{Z}_0$ of the free boundary. In particular, (1.1c) implies that the mass