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ScienceDirect
J. Differential Equations 257 (2014) 185–206
Journal of Differential Equations
www.elsevier.com/locate/jde

Spectral stability of shock waves associated with not genuinely nonlinear modes

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Received 29 October 2013
Available online 21 April 2014

Abstract

We study viscous shock waves that are associated with a simple mode (λ, r) of a system $u_t + f(u)_x = u_{xx}$ of conservation laws and that connect states on either side of an 'inflection' hypersurface Σ in state space at whose points $r \cdot \nabla\lambda = 0$ and $(r \cdot \nabla)^2\lambda \neq 0$. Such loss of genuine nonlinearity, the simplest example of which is the cubic scalar conservation law $u_t + (u^3)_x = u_{xx}$, occurs in many physical systems. We show that such shock waves are spectrally stable if their amplitude is sufficiently small. The proof is based on a direct analysis of the eigenvalue problem by means of geometric singular perturbation theory. Well-chosen rescalings are crucial for resolving degeneracies. By results of Zumbrun the spectral stability shown here implies nonlinear stability of these shock waves.

MSC: 35L67; 35B35; 34E13

Keywords: Viscous shock waves; Spectral stability; Evans function; Geometric singular perturbation theory

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http://dx.doi.org/10.1016/j.jde.2014.03.018

1. Introduction

This paper is on viscous shock waves

$$u(x,t) = \phi(x - st), \qquad \phi(\pm\infty) = u^\pm \in U, \tag{1}$$

in strictly hyperbolic systems of conservation laws

$$u_t + f(u)_x = u_{xx}$$

with state space U, a convex, open subset of $\mathbb{R}^n$, and smooth flux $f : U \to \mathbb{R}^n$. Let

$$\lambda_1(u) < \cdots < \lambda_n(u), \quad u \in U,$$

denote the eigenvalues of $Df(u)$ with associated smooth right eigenvectors $r_j(u)$,

$$Df(u)r_j(u) = \lambda_j(u)r_j(u), \quad j = 1, \dots, n.$$

We consider the case of a not genuinely nonlinear mode [16]. More precisely, we assume that the flux possesses a hypersurface $\Sigma \subset U$ such that, for a certain $k \in \{1, \dots, n\}$,

$$r_k \cdot \nabla\lambda_k(u) = 0, \qquad (r_k \cdot \nabla)^2\lambda_k(u) > 0, \quad u \in \Sigma. \tag{2}$$

The hypersurface Σ has the same meaning for the mode (λ_k, r_k) as an inflection point of the flux function does for a scalar conservation law: Along an integral curve of the vector field r_k passing through Σ, the characteristic speed λ_k is locally decreasing 'before' Σ, minimal at the intersection point, and increasing 'after' Σ. The shock waves (1) of interest are k-shocks in the sense of Lax, i.e.,

$$\lambda_{k-1}(u^-) < s < \lambda_k(u^+) \quad \text{and} \quad \lambda_k(u^-) < s < \lambda_{k+1}(u^+),$$

whose end states u^- and u^+ lie on different sides of Σ. The purpose of this paper is to show that these shock waves are spectrally stable, at least if their amplitude $|u^+ - u^-|$ is sufficiently small.

Physics provides many examples of not genuinely nonlinear modes (2). They are standard, e.g. in elasticity [2], in magnetohydrodynamics [5], and also in simple retrograde fluids [21,13]. It would be interesting to extend the results of this paper to these systems with the corresponding physical viscosities.

For any Laxian shock wave (1), spectral stability can be formulated as a property of the Evans function, as we now recapitulate; for details, we refer to [1,9,23,6]. The object of investigation is the non-autonomous eigenvalue problem

$$W' = \begin{pmatrix} Df(\phi) - sI & I \\ \kappa I & 0 \end{pmatrix} W, \tag{3}$$

on $\mathbb{C}^{2n}$, in which $\kappa \in \mathbb{C}$ is the spectral parameter; the differentiation is with respect to $\xi = x - st$. An eigenfunction associated with the eigenvalue κ is a solution W of (3) with $W(-\infty) = 0 = W(+\infty)$. Due to the shift invariance of the profile, $W = (\phi', 0)$ is an eigenfunction associated