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Existence and separation of positive radial solutions for semilinear elliptic equations

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Abstract

We consider the semilinear elliptic equation $\Delta u + K(|x|)u^p = 0$ in $\mathbf{R}^N$ for $N > 2$ and $p > 1$, and study separation phenomena of positive radial solutions. With respect to intersection and separation, we establish a classification of the solution structures, and investigate the structures of intersection, partial separation and separation. As a consequence, we obtain the existence of positive solutions with slow decay when the oscillation of the function $r^{-\ell}K(r)$ with $\ell > -2$ around a positive constant is small near $r = \infty$ and p is sufficiently large. Moreover, if the assumptions hold in the whole space, the equation has the structure of separation and possesses a singular solution as the upper limit of regular solutions. We also reveal that the equation changes its nature drastically across a critical exponent p_c which is determined by N and the order of the behavior of $K(r)$ as $r = |x| \to 0$ and ∞. In order to understand how subtle the structure is on K at $p = p_c$, we explain the criticality in a similar way as done by Ding and Ni (1985) [6] for the critical Sobolev exponent $p = (N + 2)/(N - 2)$.

Keywords: Semilinear elliptic equations; Entire solution; Separation; Partial separation; Singular solution; Asymptotic behavior

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1. Introduction

In this paper, we study positive solutions to the semilinear elliptic equation

$$\Delta u + K\bigl(|x|\bigr)u^p = 0, \quad x \in \mathbf{R}^N \setminus \{0\},$$

where $N > 2$, $p > 1$, and $K(r)$ is a continuous function on $(0, \infty)$. In particular, we are interested in radially symmetric solutions of the form $u = u(r)$ with $r = |x|$, which satisfy the ordinary differential equation

$$u'' + \frac{N-1}{r}u' + K(r)u^p = 0, \quad r > 0. \tag{1.1}$$

Throughout this paper, we assume that $K \in C(0, \infty)$ satisfies $K(r) \geq 0$ and $K(r) \not\equiv 0$ for $r \in (0, \infty)$, and

$$\int_0 rK(r)dr < \infty \quad \text{and} \quad r^2K(r) \to 0 \quad \text{as } r \to 0. \tag{1.2}$$

Under these conditions, it follows by a standard argument that for each $\alpha > 0$, Eq. (1.1) with $u(0) = \alpha$ has a unique local positive solution $u \in C[0, \epsilon) \cap C^2(0, \epsilon)$ for some $\epsilon > 0$. From now on, we denote by $u_\alpha(r)$ the solution of (1.1) with $u(0) = \alpha$.

A typical example is that $K(r) = r^\ell$ with $\ell > -2$, i.e.,

$$u'' + \frac{N-1}{r}u' + r^\ell u^p = 0, \quad r > 0. \tag{1.3}$$

It is known in [14] that u_α is positive on $[0, \infty)$ for all $\alpha > 0$ if $p \geq (N + 2 + 2\ell)/(N - 2)$. One of the qualitative questions is whether two solutions with different initial values intersect or not. The subject has lately attracted considerable attention since the structure of separation is closely related with the stability of positive stationary solutions of the corresponding parabolic equation. In fact, Wang in [17] showed that any two positive solutions of (1.3) cannot intersect each other if and only if $p \geq p_c(\ell)$, where

$$p_c(\ell) = \begin{cases} \frac{(N-2)^2-2(\ell+2)(N+\ell)+2(\ell+2)\sqrt{(N+\ell)^2-(N-2)^2}}{(N-2)(N-10-4\ell)} & \text{if } N > 10 + 4\ell \\ \infty & \text{if } N \leq 10 + 4\ell. \end{cases}$$

When $p \geq p_c(\ell)$, it turns out that the steady states of (1.3) are stable in suitable weighted uniform norms. The asymptotic behavior at ∞ suggests the topology. See, e.g., [7,8]. In the case $\ell = 0$, that is, $K \equiv 1$, this exponent is known as Joseph–Lundgren exponent [10] and has appeared in studies of [10,7,8].

In order to consider the phenomena for a general class of K, we first summarize what conditions lead to the existence of positive entire solutions of (1.1). It is known by [12,14] that, if $p \geq (N + 2 + 2\ell)/(N - 2)$ with $\ell > -2$, and

$$\frac{d}{dr}\bigl(r^{-\ell}K(r)\bigr) \leq 0 \quad \text{for all } r > 0, \tag{1.4}$$