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Algebraic and analytical tools for the study of the period function

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Abstract

In this paper we consider analytic planar differential systems having a first integral of the form $H(x, y) = A(x) + B(x)y + C(x)y^2$ and an integrating factor $\kappa(x)$ not depending on y. Our aim is to provide tools to study the period function of the centers of this type of differential system and to this end we prove three results. Theorem A gives a characterization of isochronicity, a criterion to bound the number of critical periods and a necessary condition for the period function to be monotone. Theorem B is intended for being applied in combination with Theorem A in an algebraic setting that we shall specify. Finally, Theorem C is devoted to study the number of critical periods bifurcating from the period annulus of an isochrone perturbed linearly inside a family of centers. Four different applications are given to illustrate these results.

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1. Introduction

The present paper is concerned with the period function of centers. A critical point p of a planar differential system is a *center* if it has a punctured neighborhood that consists entirely of periodic orbits surrounding p. The largest punctured neighborhood with this property is called

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the *period annulus* of the center and, in what follows, it will be denoted by $\mathscr{P}$. The *period function* of the center assigns to each periodic orbit in $\mathscr{P}$ its period. If the period function is constant, then the center is said to be *isochronous*. Since the period function is defined on the set of periodic orbits in $\mathscr{P}$, in order to study its qualitative properties usually the first step is to parameterize this set. This can be done, for instance, by taking a transversal section on $\mathscr{P}$ or, in case that the differential system has a global first integral, by using the energy level of the periodic orbits. If $\{\gamma_s\}_{s\in(0,1)}$ is such a parameterization, then $s \longmapsto T(s) := \{\text{period of } \gamma_s\}$ is a smooth map that provides the qualitative properties of the period function that we are concerned about. In particular the existence of *critical periods*, which are isolated critical points of this function, i.e., $\hat{s} \in (0,1)$ such that $T'(s) = \alpha(s-\hat{s})^k + \mathrm{o}((s-\hat{s})^k)$ with $\alpha \neq 0$ and $k \geqslant 1$. In this case we shall say that $\gamma_{\hat{s}}$ is a *critical periodic orbit* of multiplicity k of the center. One can readily see that the number, character (maximum or minimum) and distribution of the critical periods do not depend on the particular parameterization of the set of periodic orbits used.

Questions related to the behavior of the period function have been extensively studied. Let us quote, for instance, the problems of isochronicity (see [9,11,25]), monotonicity (see [5,33,34]) or bifurcation of critical periods (see [6,10,20]). These are in fact subproblems of a more general question that asks for a bound (if any) on the number of critical periods that a given family of centers can have. In this regard the most studied family is the quadratic one. According to Chicone's conjecture, see [6,7], if a quadratic system has a center with a non-monotonic period function then, by an affine transformation and a constant rescaling of time, it can be transformed to the Loud normal form and, in this case, it has at most two critical periods. This conjecture is the analogue for the period function of the second part of *Hilbert's 16th problem* for quadratic systems (see [31]), which asks for a bound on the number of limit cycles that these systems can have. This similarity is not only conceptual but also in the techniques used and in the degree of difficulty. Chicone's conjecture has attracted interest of many authors (see [8,12,14,16,23,24,27,34,35] and references therein) and there is much analytic evidence that it is true. The majority of the results are concerned about monotonicity because there is a lack of tools to investigate centers with non-monotonic period function. To our knowledge the only exception to this are [23,24,36,37], which succeed in studying quadratic centers with non-monotonic period function by showing that it verifies a Picard–Fuchs equation. The problem with this approach is that we must have a rational first integral to begin with.

The goal of the present paper is to provide tools to bound the number of critical periods of a center. Thus, Theorem A can be viewed as the continuation of the results in [29], where similar tools are developed for potential systems. We generalize them to consider a wider class of differential systems, which includes the ones in Loud normal form. The second result that we obtain, Theorem B, is addressed to an algebraic setting where Theorem A is particularly useful. Finally, Theorem C is devoted to bound the number of critical periods bifurcating from the period annulus of an isochronous center. Let us note in this regard that in our opinion it is lacking an accurate definition of this notion in the literature. In this paper we propose one, see Definition 2.3, which is in fact more general because it can also be applied to the critical periods bifurcating from a center or a polycycle.

The organization of the paper is as follows. Section 2 is devoted to introduce the definitions of the notions used henceforth and to state the main results. Then, after proving some preliminary lemmas, we show Theorems A, B and C in Section 3. In Section 4 we give four different applications of these results. More precisely, Proposition 4.1 constitutes a new result, whereas in Propositions 4.2, 4.3 and 4.4 we revisit already known results with the aim of showing the simplicity of our tools in relation to the previous approaches.