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An explicit solution of Burgers equation with stationary point source

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Abstract

Existence, uniqueness and regularity of the global weak solution to the Burgers equation with a reaction term is shown when the reaction term is given as a time independent point source and produces heat constantly. An explicit solution is obtained and used to show the long time asymptotic convergence of the solution to a steady state. For the heat equation case without any convection the solution diverges everywhere as time increases and hence it is the first order convection term that gives the compactness of the solution trajectory of the Burgers equation with reaction.

Keywords: Cole–Hopf transformation; Heat equation; Long time asymptotics

1. Introduction

In many occasions a solution to an elliptic equation is understood as the longtime asymptotic limit of a solution to a parabolic equation. A study of such a convergence provides a good chance to understand the connection between the two groups of partial differential equations. However,

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there are subtle issues in studying such longtime asymptotics, which are overlooked in many cases. The purpose of this paper is to develop an explicit example to understand such subtle issues related to the roles of advection and diffusion. Consider the heat equation with a positive heat source:

$$\begin{aligned} u_t - u_{xx} &= \delta, \quad x \in \mathbf{R},\ 0 < t, \\ u(x,0) &= u_0(x), \quad x \in \mathbf{R}, \end{aligned} \tag{1.1}$$

where the heat source $\delta = \delta(x)$ is the time independent Dirac delta measure and the initial data u_0 is in $L^1(\mathbf{R})$. Notice that it is not the initial value, but the source that is remembered by the elliptic limit of the parabolic problem. Formally, one can see that the total heat energy increases constantly, i.e.,

$$\frac{d}{dt}\int u(x,t)dx = \int \delta(x)dx = 1.$$

Therefore, the solution does not converge in $L^1(\mathbf{R})$ as $t \to \infty$. Then, can we expect a pointwise convergence?[2]

Intuitively one might guess that as $t \to \infty$ the solution u would approach one of the steady states. Here, a steady state solution, say ω, of course satisfies

$$-\omega_{xx} = \delta$$

so that it can be written as a sum of the fundamental solution of Laplace's equation and a harmonic function $h(x)$:

$$\omega(x) = -|x|/2 + h(x). \tag{1.2}$$

However, this guess for the asymptotic behavior as $t \to \infty$ is wrong. Observe the solution u, which is explicitly given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} e^{-y^2/4t} u_0(x-y)\, dy + \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-x^2/4(t-\tau)}\, d\tau.$$

The first term is from the initial heat distribution and vanishes as $t \to \infty$. The second term is from the inhomogeneous heat source and equals $\sqrt{t/\pi}$ at $x = 0$, which diverges to $+\infty$ with order $O(\sqrt{t})$ as $t \to \infty$. If $x \neq 0$, by introducing $\xi = x^2/4(t-\tau)$, the second term is written as

$$\frac{|x|}{4\sqrt{\pi}} \int_{x^2/4t}^{\infty} \xi^{-3/2} e^{-\xi}\, d\xi.$$

[2] The answer depends on the space dimensions. For example, the answer is affirmative if the space dimension is $n \geq 3$.