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J. Differential Equations 257 (2014) 2543–2564

Journal of Differential Equations

www.elsevier.com/locate/jde

Global solution curves for self-similar equations

Philip Korman

Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221-0025, United States

Received 14 October 2013; revised 27 May 2014

Available online 10 June 2014

Abstract

We consider positive solutions of a semilinear Dirichlet problem

$$\Delta u + \lambda f(u) = 0, \quad \text{for } |x| < 1, \qquad u = 0, \quad \text{when } |x| = 1$$

on a unit ball in R^n. For four classes of self-similar equations it is possible to parameterize the entire (global) solution curve through the solution of a single initial value problem. This allows us to derive results on the multiplicity of solutions, and on their Morse indices. In particular, we easily recover the classical results of D.D. Joseph and T.S. Lundgren [6] on the Gelfand problem. Surprisingly, the situation turns out to be different for the generalized Gelfand problem, where infinitely many turns are possible for any space dimension $n \geq 3$. We also derive detailed results for the equation modeling electrostatic micro-electromechanical systems (MEMS), in particular we easily recover the main result of Z. Guo and J. Wei [4], and we show that the Morse index of the solutions increases by one at each turn. We also consider the self-similar Henon's equation.

MSC: 35J60; 35B40

Keywords: Parameterization of the global solution curves; Infinitely many solutions; Morse indices; The Gelfand problem

http://dx.doi.org/10.1016/j.jde.2014.05.045

0022-0396/

1. Introduction

We consider radial solutions on a ball in R^n for four special classes of equations, the ones self-similar under scaling. For example, consider the so called Gelfand equation ($u = u(x)$, $x \in R^n$)

$$\Delta u + \lambda e^u = 0, \quad \text{for } |x| < 1, \qquad u = 0, \quad \text{when } |x| = 1. \tag{1.1}$$

Here λ is a positive parameter. By the maximum principle, solutions of (1.1) are positive, and then by the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [2] they are radially symmetric, i.e., $u = u(r)$, $r = |x|$, and it satisfies

$$u'' + \frac{n-1}{r}u' + \lambda e^u = 0, \quad \text{for } 0 < r < 1, \qquad u'(0) = u(1) = 0. \tag{1.2}$$

This theorem also asserts that $u'(r) < 0$ for all $0 < r < 1$, which implies that the value of $u(0)$ gives the L^∞ norm of our solution. Moreover, $u(0)$ is a *global parameter*, i.e., it uniquely identifies the solution pair $(\lambda, u(r))$, see e.g., P. Korman [9]. It follows that a two-dimensional curve $(\lambda, u(0))$ completely describes the solution set of (1.1). The change of variables $v = u + a$, $\xi = br$, with constant a and b will transform the equation in (1.2) into the same equation if $e^a = b^2$. Here is what this self-similarity "buys" us. Let $w(t)$ be the solution of the following initial value problem

$$w'' + \frac{n-1}{t}w' + e^w = 0, \qquad w(0) = 0, \qquad w'(0) = 0 \quad (t > 0), \tag{1.3}$$

which is easily seen to be negative, and defined for all $t \in (0, \infty)$. It turns out that $w(t)$ gives us the entire solution curve of (1.2) (or of (1.1)):

$$\big(\lambda, u(0)\big) = \big(t^2 e^{w(t)}, -w(t)\big), \tag{1.4}$$

parameterized by $t \in (0, \infty)$. In particular, $\lambda = \lambda(t) = t^2 e^{w(t)}$, and

$$\lambda'(t) = te^w\big(2 + tw'\big),$$

so that the solution curve travels to the right (left) in the $(\lambda, u(0))$ plane if $2 + tw' > 0$ (< 0). This makes us interested in the roots of the function $2 + tw'$. If we set this function to zero

$$2 + tw' = 0,$$

then solution of this equation is of course $w(t) = a - 2\ln t$. Amazingly, if we choose $a = \ln(2n - 4)$, $n \ge 3$, then

$$w_0(t) = \ln(2n - 4) - 2\ln t$$