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Complete entrainment of Kuramoto oscillators with inertia on networks via gradient-like flow

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Abstract

We study the asymptotic complete entrainment of Kuramoto oscillators with inertia on symmetric and connected network. We provide several sufficient conditions for the asymptotic complete entrainment in terms of initial phase-frequency configurations, strengths of inertia and coupling, and natural frequency distributions. For this purpose, we reinterpret the Kuramoto oscillators with inertia as a second-order gradient-like flow, and adopt analytical methods based on several Lyapunov functions to apply the convergence estimate studied by Haraux and Jendoubi [21]. Our approach does not require any spectral information of the graph associated with the given network structure.

Keywords: Kuramoto oscillators; Inertia; Network; Complete entrainment; Gradient-like flow

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1. Introduction

Consider a symmetric and connected network, which can be realized with a weighted graph $G = (V, E, \Psi)$. Here, $V = \{u_1, \ldots, u_N\}$ and $E \subseteq V \times V$ are vertex and edge sets, respectively, and $\Psi = (\psi_{ij})$ is an $N \times N$ matrix whose element ψ_{ij} denotes the capacity of the edge (communication weight) flowing from u_j to u_i. For a given network or graph G, we assume that Kuramoto oscillators are located at the vertices of the graph, and interactions between them are represented by E and Ψ. Let $\theta_i = \theta_i(t) \in \mathbb{R}$ be the phase of the i-th oscillator with natural frequency Ω_i. Then, the dynamics of Kuramoto oscillators with inertia is governed by the second-order system:

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + K \sum_{j=1}^{N} \psi_{ij} \sin(\theta_j - \theta_i), \quad i = 1, 2, \ldots, N, \tag{1.1}$$

subject to the initial data:

$$(\theta_i, \omega_i)(0) = (\theta_{i0}, \omega_{i0}), \tag{1.2}$$

where $m > 0$ and $K > 0$ represent the strengths of inertia and coupling, respectively, and $\omega_i = \dot{\theta}_i$ is the phase velocity (instantaneous frequency). We assume that the capacity matrix $\Psi = (\psi_{ij})$ is symmetric and connected:

(i) $\psi_{ij} = \psi_{ji} \geq 0$, $1 \leq i, j \leq N$.
(ii) For any $(u_i, u_j) \in V \times V$, there is a shortest path from u_i to u_j, say

$$u_i = u_{k_0} \to u_{k_1} \to \cdots \to u_{k_{d_{ij}}} = u_j, \quad (u_{k_l}, u_{k_{l+1}}) \in E, \ l = 0, 1, \ldots, d_{ij} - 1. \tag{1.3}$$

Here, d_{ij} denotes the distance between nodes u_i and u_j, i.e., the length of the shortest path from node u_i to node u_j.

The system (1.1) with all-to-all coupling $\psi_{ij} = \frac{1}{N}$ was first introduced by Ermentrout [16] as a phenomenological model to explain the slow synchronization of certain biological systems [1,5,35,36], e.g., fireflies of the Pteroptyx malaccae. This model also appears as a part of the dynamical system describing the dynamics of the array of superconducting Josephson junctions [15,37–40]. The second-order phase model (1.1) has been investigated in several studies [2,17,22], and it exhibits rich phenomena such as discontinuous first-order phase transition and hysteresis [32,33], when the inertia strength is sufficiently large, compared to the Kuramoto oscillators (1.1) with $m = 0$. We refer to [1,4,12,24,25,27–29,31] for the mathematical results of the globally coupled Kuramoto model, and [3,30] for the synchronization in networks. Most studies have considered the all-to-all case, and to the best of the authors' knowledge, the dynamics of locally coupled Kuramoto oscillators with inertia has not been addressed in the previous studies. Hence the main objective of this paper is to investigate the relation between the asymptotic complete entrainment in Definition 1.1 and stationary network structure.

Next, we recall some definitions of asymptotic behaviors for Kuramoto oscillators as follows.