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Multiplicity of positive and nodal solutions for scalar field equations

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Abstract

In this paper the question of finding infinitely many solutions to the problem $-\Delta u + a(x)u = |u|^{p-2}u$, in $\mathbb{R}^N$, $u \in H^1(\mathbb{R}^N)$, is considered when $N \geq 2$, $p \in (2, 2N/(N-2))$, and the potential $a(x)$ is a positive function which is not required to enjoy symmetry properties. Assuming that $a(x)$ satisfies a suitable "slow decay at infinity" condition and, moreover, that its graph has some "dips", we prove that the problem admits either infinitely many nodal solutions or infinitely many constant sign solutions. The proof method is purely variational and allows to describe the shape of the solutions.

1. Introduction

This paper deals with the question of the existence of multiple solutions for equations of the type

$$-\Delta u + a(x)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N, \tag{E}$$

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where $N \geq 2$, $p > 1$, $p < 2^* - 1 = \frac{N+2}{N-2}$ if $N \geq 3$, and the potential $a : \mathbb{R}^N \to \mathbb{R}$ is a positive smooth function.

Eq. (E) has strongly attracted the researchers' attention and has been extensively studied because it appears naturally in the study of problems coming from applied sciences. Actually, Euclidean scalar field equations like (E) arise in several contexts: the most known is probably the search of certain kind of solitary waves in nonlinear equations of the Klein–Gordon or Schrödinger type, but also questions in nonlinear optics, condensed matter physics, laser propagation, constructive field theory lead to look for solutions of equations like (E) (see e.g. [7,10]).

However, it is worth pointing out that not the least reason of interest for (E) rests on some of its mathematical features that give rise to challenging difficulties. Eq. (E) is variational, so it is reasonable to search its solutions as critical points of the corresponding "action" functional defined in $H^1(\mathbb{R}^N)$ by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + a(x)u^2\right) dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx,$$

but a lack of compactness, due to the unboundedness of the domain $\mathbb{R}^N$, prevents a straight application of the usual variational methods. Whatever p is, the embedding $j : H^1(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is not compact, hence arguments like those based on the Palais–Smale condition can fail.

When the potential $a(x)$ enjoys radial symmetry the difficulty in treating (E) can be overcome working in the subspace $H_r(\mathbb{R}^N)$ of $H^1(\mathbb{R}^N)$ consisting of radially symmetric functions. $H_r(\mathbb{R}^N)$ embeds compactly in $L^p(\mathbb{R}^N)$, $p \in (2, 2^*)$, so standard variational techniques work and, as for bounded domains, one can show [29,7,19,30] that (E) possesses a positive radial solution and infinitely many changing sign radial solutions. Furthermore, the radial symmetry of the potential and, of course, of $\mathbb{R}^N$ plays a basic role also when one is looking for non-radially symmetric solutions (see e.g. [6,22,31]). For other multiplicity results, obtained under either radial or lighter symmetry assumptions, see e.g. [12,16,17,25] and references there.

On the contrary, when $a(x)$ has no symmetry properties there is no way to avoid the compactness question. Moreover, the difficulty is not only a technical fact, as one can understand considering (E) with a potential increasing along a direction, then, as shown in [13], (E) has only the trivial solution. Nevertheless, some considerable progress has been performed also for non-symmetric potentials, even if the situation is still not completely clear and the picture is fragmentary. Most of researches have been concerned with the case in which

$$\lim_{|x| \to \infty} a(x) = a_\infty > 0.$$

Under the above assumption various existence and multiplicity results for (E) have been stated, but it must be remarked at once that the topological situation appears quite different according to that $a(x)$ approaches a_∞ from below or from above. In the first case the existence of a least energy positive solution can be obtained by minimization, using concentration-compactness type arguments (see [21,28]). In the latter, the functional I may not attain its infimum on the natural constraint $I'(u)[u] = 0$ and critical points of I have to be searched working at higher energy levels and using more subtle variational and topological tools. By this kind of arguments, in [5] and [4], the existence of a positive solution to (E) has been proved under a suitable "fast decay" assumption on $a(x)$.