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# Determining the first order perturbation of a bi-harmonic operator on bounded and unbounded domains from partial data

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**Abstract**

In this paper we study inverse boundary value problems with partial data for the bi-harmonic operator with first order perturbation. We consider two types of subsets of $\mathbb{R}^n$ ($n \geq 3$), one is an infinite slab, the other is a bounded domain. In the case of a slab, we show that, from Dirichlet and Neumann data given either on the different boundary hyperplanes of the slab or on the same boundary hyperplane, one can uniquely determine the magnetic potential and the electric potential.

In the case of a bounded domain, we show the unique determination of the magnetic potential and the electric potential from partial Dirichlet and Neumann data under two different assumptions. The first assumption is that the magnetic and electric potentials are known in a neighborhood of the boundary, in this situation we obtain the uniqueness result when the Dirichlet and Neumann data are only given on two arbitrary open subsets of the boundary. The second assumption is that the Dirichlet and Neumann data are known on the same part of the boundary whose complement is a part of a hyperplane, we also establish the unique determination result in this local data case.


## 1. Introduction and statement of results

A bi-harmonic operator with first order perturbation is a differential operator of the form

$$\mathcal{L}_{A,q}(x, D) := \Delta^2 + A(x) \cdot D + q(x)$$

with $D = \frac{1}{i}\nabla$. Here $A$ is a complex-valued vector field called the magnetic potential, $q$ is a complex-valued function called the electric potential. The bi-harmonic operator arises in physics

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when considering the equilibrium configuration of an elastic plate hinged along the boundary. It is also widely used in other physical models, see [6]. In this paper we study the identifiability of the first order perturbation of a bi-harmonic operator from partial boundary measurements in two types of open subsets of $\mathbb{R}^n$, the first type is an infinite slab, and the second type is a bounded domain with $C^\infty$ boundary.

First we consider an infinite slab $\Sigma$. The geometry of an infinite slab arises in many applications, for instance, in the study of wave propagation in marine acoustics. It is also a simple geometric setting in medical imaging. By choosing appropriate coordinates, we may assume that

$$\Sigma := \{x = (x', x_n) \in \mathbb{R}^n : x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1},\ 0 < x_n < L\}, \quad L > 0.$$

Its boundary consists of two parallel hyperplanes

$$\Gamma_1 := \{x \in \mathbb{R}^n : x_n = L\} \qquad \Gamma_2 := \{x \in \mathbb{R}^n : x_n = 0\}.$$

Given $(f_1, f_2) \in H^{\frac{7}{2}}(\Gamma_1) \times H^{\frac{3}{2}}(\Gamma_1)$ with $f_1, f_2$ compactly supported on $\Gamma_1$, we are interested in the following Dirichlet problem

$$\begin{cases} \mathcal{L}_{A,q} u = 0 & \text{in } \Sigma \\ u = f_1 \quad \Delta u = f_2 & \text{on } \Gamma_1 \\ u = 0 \quad \Delta u = 0 & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

In Appendix A we show that problem (1.1) has a unique solution in $H^4(\Sigma)$, where $H^4(\Sigma)$ is the standard Sobolev space on $\Sigma$. Let $\gamma_1 \subset \Gamma_1$, $\gamma_2 \subset \partial\Sigma$ be non-empty open subsets, we define the partial Cauchy data set for the boundary value problem (1.1) as

$$\begin{aligned} C_{A,q}^{\gamma_1,\gamma_2}(\Sigma) := \{ & (u|_{\gamma_1}, \Delta u|_{\gamma_1}, \partial_\nu u|_{\gamma_2}, \partial_\nu(\Delta u)|_{\gamma_2}) : \mathcal{L}_{A,q} u = 0 \text{ in } \Sigma,\ u \in H^4(\Sigma), \\ & \operatorname{supp}(u|_{\Gamma_1}) \text{ and } \operatorname{supp}(\Delta u|_{\Gamma_1}) \text{ are compact and contained in } \gamma_1, \\ & u|_{\Gamma_2} = \Delta u|_{\Gamma_2} = 0\}, \end{aligned}$$

where $\nu$ is the unit outer normal vector field to $\partial\Sigma = \Gamma_1 \cup \Gamma_2$. Here we think of $(u|_{\gamma_1}, \Delta u|_{\gamma_1})$ as the Dirichlet data prescribed only on $\gamma_1$, and $\partial_\nu u|_{\gamma_2}, \partial_\nu(\Delta u)|_{\gamma_2}$ as the Neumann data measured on $\gamma_2$. The inverse problem we will study is as follows: for fixed $\gamma_1$ and $\gamma_2$, assuming that

$$C_{A^{(1)},q^{(1)}}^{\gamma_1,\gamma_2}(\Sigma) = C_{A^{(2)},q^{(2)}}^{\gamma_1,\gamma_2}(\Sigma),$$

can we conclude $A^{(1)} = A^{(2)}$ and $q^{(1)} = q^{(2)}$ in $\Sigma$?

We will show this is valid for some open subsets $\gamma_1, \gamma_2$, if $A^{(j)}, q^{(j)}$, $j = 1, 2$ are compactly supported in the closure $\bar{\Sigma}$. Our first result considers the case when the data and the measurements are on different boundary hyperplanes.

**Theorem 1.1.** *Let $\Sigma \subset \mathbb{R}^n$ $(n \geq 3)$ be an infinite slab with boundary hyperplanes $\Gamma_1$ and $\Gamma_2$. Let $A^{(j)} \in W^{1,\infty}(\Sigma; \mathbb{C}^n) \cap \mathcal{E}'(\bar{\Sigma}; \mathbb{C}^n)$, $q^{(j)} \in L^\infty(\Sigma; \mathbb{C}) \cap \mathcal{E}'(\bar{\Sigma}; \mathbb{C})$, $j = 1, 2$. Denote by $B \subset \mathbb{R}^n$ an open ball containing the supports of $A^{(j)}, q^{(j)}$, $j = 1, 2$. Let $\gamma_j \subset \Gamma_j$ be open sets such that $\Gamma_j \cap \bar{B} \subset \gamma_j$, $j = 1, 2$. If*