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The effective boundary conditions and their lifespan of the logistic diffusion equation on a coated body

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Abstract

We consider the logistic diffusion equation on a bounded domain, which has two components with a thin coating surrounding a body. The diffusion tensor is isotropic on the body, and anisotropic on the coating. The size of the diffusion tensor on these components may be very different; within the coating, the diffusion rates in the normal and tangent directions may be in different scales. We find effective boundary conditions (EBCs) that are approximately satisfied by the solution of the diffusion equation on the boundary of the body. We also prove that the lifespan of each EBC, which measures how long the EBC remains effective, is infinite. The EBCs enable us to see clearly the effect of the coating and ease the difficult task of solving the PDE in a thin region with a small diffusion tensor. The motivation of the mathematics includes a nature reserve surrounded by a buffer zone.

Keywords: Logistic diffusion equation; Coating; Anisotropic diffusion tensor; Effective boundary condition; Lifespan

1. Introduction

In [6,10,11], we studied the linear heat equation on a bounded domain Ω, which consists of the body Ω_1 and the surrounding thin coating Ω_2 (see Fig. 1). The physical models in mind were spacecrafts and turbine engine blades protected by thermal insulators; the mathematical goal was

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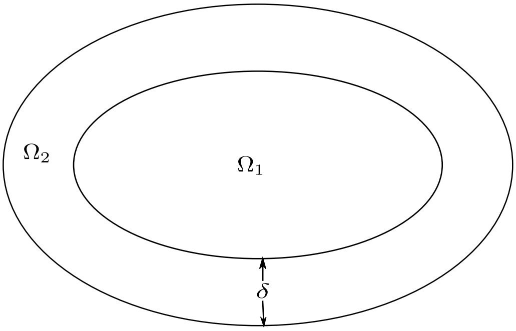

Fig. 1. $\Omega = \overline{\Omega}_1 \cup \Omega_2$. The coating Ω_2 is uniformly thick with thickness δ.

to find the "effective boundary condition" (EBC) on the boundary $\partial\Omega_1$ of the body. The EBC is approximately satisfied by the solution of the heat equation on $\partial\Omega_1$ and enables us to see the effect of the coating easily. For example, if the EBC is the Neumann boundary condition, then it means the boundary of Ω_1 is well-insulated. Furthermore, finding EBCs eases the task of numerically solving the heat equation: to solve the equation on whole Ω is a very challenging task, because of the small size of the thickness and the thermal tensor of Ω_2; but if we know the EBC on $\partial\Omega_1$, we can simply solve the heat equation on Ω_1 with the EBC, which involves no small scales.

In the present paper we initiate our program for the nonlinear case: we study the diffusion equation with the source term being logistic—the Fisher–KPP equation. The logistic term does not seem to arise in the mathematical modeling of spacecrafts and turbine engine blades, but can be used to model the overcrowding effect in the context of ecology. In that context, the "body" Ω_1 can represent a nature reserve, and the "coating" Ω_2 a buffer zone; the diffusion tensors on these two components may be very different in size and nature, for example, it can be of size $O(1)$ and isotropic on Ω_1, and of size $o(1)$ and anisotropic on Ω_2.

Let us now give the precise formulation of the problem considered in this paper. Let $\Omega_1 \subset \mathbb{R}^n$ be a bounded domain with C^2-smooth boundary; let the domain Ω_2 of thickness $\delta > 0$ be given by $\{x \notin \Omega_1 \mid 0 < \mathrm{dist}(x, \partial\Omega_1) < \delta\}$; let $\Omega = \overline{\Omega}_1 \cup \Omega_2$. Assume that Ω_1 is isotropic and Ω_2 anisotropic, so that the diffusion tensor of Ω is given by

$$A(x) = \big(a_{ij}(x)\big)_{n\times n} = \begin{cases} kI_{n\times n}, & x \in \Omega_1, \\ \sigma(\bar{a}_{ij}(x))_{n\times n}, & x \in \Omega_2, \end{cases} \tag{1.1}$$

where both k and σ are positive constants, $I_{n\times n}$ is the identity matrix and $(\bar{a}_{ij})$ is a symmetric positive definite matrix. σ is a parameter that measures the diffusion rate in all directions; if it is small, then the coating is a good insulator.

The initial boundary value problem considered in this paper is the following

$$\begin{cases} u_t = \nabla \cdot \big(A(x)\nabla u\big) + f(u), & (x,t) \in Q_T, \\ u = 0, & (x,t) \in S_T, \\ u = \varphi(x), & x \in \Omega,\ t = 0, \end{cases} \tag{1.2}$$

where $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$, $\varphi \ge 0$ and $f(u) = u(1-u)$. We first study the limit of u on Ω_1 as the thickness δ of the coating shrinks to zero. It turns out that the scaling