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# Optimal decay rate for strong solutions in critical spaces to the compressible Navier–Stokes equations

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**Abstract**

In this paper we are concerned with the convergence rates of the global strong solution to motionless state with constant density for the compressible Navier–Stokes equations in the whole space $\mathbb{R}^n$ for $n \geq 3$. It is proved that the perturbations decay in critical spaces, if the initial perturbations of density and velocity are small in $B^{\frac{n}{2}}_{2,1}(\mathbb{R}^n) \cap \dot{B}^{0}_{1,\infty}(\mathbb{R}^n)$ and $B^{\frac{n}{2}-1}_{2,1}(\mathbb{R}^n) \cap \dot{B}^{0}_{1,\infty}(\mathbb{R}^n)$, respectively.


*MSC:* 35Q30; 76N15

*Keywords:* Compressible Navier–Stokes equations; Convergence rate

## 1. Introduction

This paper studies the initial value problem for the compressible Navier–Stokes equation in $\mathbb{R}^n$:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t u + (u \cdot \nabla)u + \dfrac{\nabla P(\rho)}{\rho} = \dfrac{\mu}{\rho}\Delta u + \dfrac{\mu + \mu'}{\rho}\nabla(\nabla \cdot u), \\ (\rho, u)(0, x) = (\rho_0, u_0)(x). \end{cases} \tag{1}$$

Here $t > 0$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$; the unknown functions $\rho = \rho(t, x) > 0$ and $u = u(t, x) = (u_1(t, x), u_2(t, x), \cdots, u_n(t, x))$ denote the density and velocity, respectively; $P = P(\rho)$ is the

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pressure that is assumed to be a function of the density $\rho$; $\mu$ and $\mu'$ are the viscosity coefficients satisfying the conditions $\mu > 0$ and $\mu' + 2\mu > 0$; and $\nabla\cdot$, $\nabla$ and $\Delta$ denote the usual divergence, gradient and Laplacian with respect to $x$, respectively.

We assume that $P(\rho)$ is smooth in a neighborhood of $\bar{\rho}$ with $P'(\bar{\rho}) > 0$, where $\bar{\rho}$ is a given positive constant.

In this paper we derive the convergence rate of solution of problem (1) to the constant stationary solution $(\bar{\rho}, 0)$ as $t \to \infty$ when the initial perturbation $(\rho_0 - \bar{\rho}, u_0)$ is sufficiently small in critical spaces and $\dot{B}^0_{1,\infty}$ for $n \geq 3$.

Matsumura and Nishida [8] showed the global in time existence of solution of (1) for $n = 3$, provided that the initial perturbation $(\rho_0 - \bar{\rho}, u_0)$ is sufficiently small in $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Furthermore, the following decay estimates were obtained in [8];

$$\left\|\nabla^k(\rho - \bar{\rho}, u)(t)\right\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \quad k = 0, 1. \tag{2}$$

These results were proved by combining the energy method and the decay estimates of the semigroup $E(t)$ generated by the linearized operator $A$ at the constant state $(\bar{\rho}, 0)$.

On the other hand, Kawashita [6] showed the global existence of solution for initial perturbations sufficiently small in $H^{s_0}(\mathbb{R}^n)$ with $s_0 = [\frac{n}{2}] + 1$, $n \geq 2$. (Note that $s_0 = 2$ for $n = 3$.) Wang and Tan [11] then considered the case $n = 3$ when the initial perturbation $(\rho_0 - \bar{\rho}, u_0)$ is sufficiently small in $H^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, and proved the decay estimates (2). Okita [10] showed that if $n \geq 2$ then the following estimates hold for the solution $(\rho, u)$ of (1):

$$\left\|\nabla^k(\rho - \bar{\rho}, u)(t)\right\|_{L^2} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \quad k = 0, \cdots, s_0,$$

provided that $(\rho_0 - \bar{\rho}, u_0)$ is sufficiently small in $H^{s_0}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $s_0 = [\frac{n}{2}] + 1$. This result was shown by decomposition of the perturbation into low and high frequency parts. Moreover Li and Zhang [7] showed the density and momentum converge at the rates $(1+t)^{\frac{3}{4}-\frac{s}{2}}$ in the $L^2$-norm, when initial perturbation is sufficiently small in $H^l(\mathbb{R}^3) \cap \dot{B}^{-s}_{1,\infty}(\mathbb{R}^3)$ with $l \geq 4$ and $s \in [0, 1]$. Note that $L^1$ is included in $\dot{B}^0_{1,\infty}$.

Danchin [2] proved the global existence in critical homogeneous Besov space, which is stated as follows.

**Proposition 1.1.** (See Danchin [2].) *Let $n \geq 2$. There are two positive constants $\epsilon_1$ and $M$ such that for all $(\rho_0, u_0)$ with $(\rho_0 - \bar{\rho}) \in \dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}$, $u_0 \in \dot{B}^{\frac{n}{2}-1}_{2,1}$ and*

$$\|\rho_0 - \bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \leq \epsilon_1, \tag{3}$$

*then problem (1) has a unique global solution $(\rho, u) \in C(\mathbb{R}^+; \dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}) \times (L^1(\mathbb{R}^+; \dot{B}^{\frac{n}{2}+1}_{2,1}) \cap C(\mathbb{R}^+; \dot{B}^{\frac{n}{2}-1}_{2,1}))$ that satisfies the estimate*

$$\begin{aligned} &\sup_{t\in\mathbb{R}^+}\left\|(\rho - \bar{\rho})(t)\right\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \sup_{t\in\mathbb{R}^+}\left\|u(t)\right\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \int_0^\infty \left\|u(t)\right\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} dt \\ &\quad \leq M\big(\|\rho_0 - \bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\big). \end{aligned}$$