Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 257 (2014) 3868–3886
Journal of Differential Equations
www.elsevier.com/locate/jde

Estimates for eigenvalues of the Paneitz operator ☆

Qing-Ming Cheng

Department of Applied Mathematics, Faculty of Sciences, Fukuoka University, Fukuoka 814-0180, Japan

Received 16 April 2014; revised 17 June 2014

Available online 7 August 2014

Abstract

For an n-dimensional compact submanifold M^n in the Euclidean space $\mathbf{R}^N$, we study estimates for eigenvalues of the Paneitz operator on M^n. Our estimates for eigenvalues are sharp.

MSC: 53C40; 58C40

Keywords: A Paneitz operator; Q-curvature; Eigenvalues; The first eigenfunction

1. Introduction

For compact Riemann surfaces M^2, Li and Yau [11] introduced the notion of conformal volume, which is a global invariant of the conformal structure. They determined the conformal volume for a large class of Riemann surfaces, which admit minimal immersions into spheres. In particular, they proved that for a compact Riemann surface M^2, if there exists a conformal map from M^2 into the unit sphere $S^N(1)$, then the first eigenvalue λ_1 of the Laplacian satisfies

$$\lambda_1 \operatorname{vol}\big(M^2\big) \le 2V_c\big(N, M^2\big)$$

and the equality holds only if M^2 is a minimal surface in $S^N(1)$, where $V_c(N, M^2)$ is the conformal volume of M^2.

☆ Research partially supported by JSPS Grant-in-Aid for Scientific Research (B) No. 24340013 and Challenging Exploratory Research No. 25610016.

E-mail address: cheng@fukuoka-u.ac.jp.

http://dx.doi.org/10.1016/j.jde.2014.07.012

For 4-dimensional compact Riemannian manifolds, Paneitz [13] introduced a fourth order operator P_g defined by, letting div be the divergence for the metric g,

$$P_g f = \Delta^2 f - \operatorname{div}\left[\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)\nabla f\right], \tag{1.1}$$

for smooth functions f on M^4, where Δ and ∇ denote the Laplacian and the gradient operator with respect to the metric g on M^4, respectively, and R and Ric are the scalar curvature and Ricci curvature tensor with respect to the metric g on M^4. Furthermore, Branson [1] has generalized the Paneitz operator to an n-dimensional Riemannian manifold. For an n-dimensional Riemannian manifold (M^n, g), the operator P_g is defined by

$$P_g f = \Delta^2 f - \operatorname{div}\big[(a_n Rg + b_n \mathrm{Ric})\nabla f\big] + \frac{n-4}{2} Qf, \tag{1.2}$$

where

$$Q = c_n |\mathrm{Ric}|^2 + d_n R^2 - \frac{1}{2(n-1)}\Delta R$$

is called Q-curvature with respect to the metric g,

$$a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}, \qquad b_n = -\frac{4}{n-2},$$
$$c_n = -\frac{2}{(n-2)^2}, \qquad d_n = \frac{n(n-2)^2-16}{8(n-1)^2(n-2)^2}.$$

This operator P_g is also called Paneitz operator or Branson–Paneitz operator. It is known that Paneitz operator is conformally invariant of bi-degree $(\frac{n-4}{2}, \frac{n+4}{2})$, that is, under conformal transformation of Riemannian metric $g = e^{2w} g_0$, the Paneitz operator P_g changes into

$$P_g f = e^{-\frac{n+4}{2}w} P_{g_0}\big(e^{\frac{n-4}{2}w} f\big). \tag{1.3}$$

Let $\mathfrak{M}(M^n)$ be the set of Riemannian metrics on M^n. For each $g \in \mathfrak{M}(M^n)$, the total Q-curvature for g is defined by

$$Q[g] = \int_{M^n} Q\,dv.$$

When $n = 4$, from the Gauss–Bonnet–Chern theorem for dimension 4, we have

$$Q[g] = -\frac{1}{4}\int_{M^4} |W|^2 dv + 8\pi^2 \chi\big(M^4\big), \tag{1.4}$$

where W is the Weyl conformal curvature tensor and $\chi(M^4)$ is the Euler characteristic of M^4. Hence, we know that the total Q-curvature is a conformal invariant for dimension 4.