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J. Differential Equations 257 (2014) 1168–1193

Journal of Differential Equations

www.elsevier.com/locate/jde

The critical problem of Kirchhoff type elliptic equations in dimension four

Daisuke Naimen

Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto Sumiyoshi-ku, Osaka-shi, Osaka 558-8585, Japan

Received 24 December 2013; revised 7 April 2014

Available online 20 May 2014

Abstract

We study the following Kirchhoff type elliptic problem,

$$\begin{cases} -\left(a+b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda u^q + \mu u^3, \quad u>0 \quad \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^4$ is a bounded domain with smooth boundary $\partial\Omega$. Moreover, we assume $a, \lambda, \mu > 0$, $b \geq 0$ and $1 \leq q < 3$. In this paper, we prove the existence of solutions of (P). Our tools are the variational method and the concentration compactness argument for PS sequences.

Keywords: Kirchhoff; Nonlocal; Elliptic; Critical; Variational method

E-mail address: d12sax0J51@ex.media.osaka-cu.ac.jp.

http://dx.doi.org/10.1016/j.jde.2014.05.002
0022-0396/

1. Introduction

We investigate a Kirchhoff type elliptic problem,

$$\begin{cases} -\left(a+b\int\limits_{\Omega}|\nabla u|^2dx\right)\Delta u=\lambda u^q+\mu u^3 & \text{in } \Omega, \\ u>0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega\subset\mathbb{R}^4$ is a bounded domain with smooth boundary $\partial\Omega$. We assume $a,\lambda,\mu>0$, $b\geq 0$ and $1\leq q<3$. In this paper, we prove the existence of solutions of (P).

Our problem (P) describes the stationary state of the Kirchhoff type quasilinear hyperbolic equation such as

$$\frac{\partial^2 u}{\partial t^2}-M\left(\int\limits_{\Omega}|\nabla u|^2dx\right)\Delta u=f(x,t,u), \tag{P$_0$}$$

where $M:\mathbb{R}^+\to\mathbb{R}^+$ is some function. (P$_0$) appears in the theory of the nonlinear vibrations on physics [15]. The solvability of (P$_0$) is also discussed on mathematics [6,7,9,14,24] etc. We can refer to the survey [1].

In recent years, the analysis of the stationary problems of (P$_0$) has been extensively carrying out by many authors, see [2–4,10,11,16–18,20–22,26,28,29] and so on. By them, several existence results are successfully obtained via the variational and topological methods even for the critical case. But most of them treat only three or less dimensional case except for [3,10,18]. Here we emphasize that we would treat the 4-dimensional critical problem (P). In our case, a typical difficulty occurs in proving the existence of solutions. It is caused by the lack of the compactness of the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^4(\Omega)$. Furthermore, in view of the corresponding energy, the interaction between the Kirchhoff type perturbation $\|u\|^4_{H_0^1(\Omega)}$ and the critical nonlinearity $\int_\Omega u^4dx$ is crucial. In the following, we can see the effect of such an interaction on the existence. To our best knowledge, this paper is the first one which essentially attacks the Brezis–Nirenberg problem for 4-dimensional Kirchhoff type equations. Lastly we remark that if we take $N\geq 5$, the critical Sobolev exponent $2^*:=2N/(N-2)$ is strictly less than 4. Thus the treatments for the higher dimensional problem must be different from those in this paper. This is because our ideas are strictly based on the fact that the critical exponent is just equal to the exponent on the Kirchhoff term $\|u\|^4$. How to get the solvability of the higher dimensional problem is another and interesting problem for our future. We will again put a remark on the higher dimensional problem in Section 4.

1.1. Statement of results

Firstly we consider the cases $q=1$. Let $S>0$ be the usual Sobolev constant defined by

$$\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\int_\Omega|\nabla u|^2dx}{(\int_\Omega u^4dx)^{1/2}},$$

and $\lambda_1>0$ be the principal eigenvalue of $-\Delta$ on Ω. Our result is the following.