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Iteration theory of L-index and multiplicity of brake orbits

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Abstract

In this paper, we first establish the Bott-type iteration formulas and some abstract precise iteration formulas of the Maslov-type index theory associated with a Lagrangian subspace for symplectic paths. As an application, we prove that there exist at least $[\frac{n}{2}]+1$ geometrically distinct brake orbits on every C^2 compact convex symmetric hypersurface Σ in $\mathbf{R}^{2n}$ satisfying the reversible condition $N\Sigma=\Sigma$. Furthermore, if all brake orbits on this hypersurface are nondegenerate, then there are at least n geometrically distinct brake orbits on it. As a consequence, we show that there exist at least $[\frac{n}{2}]+1$ geometrically distinct brake orbits in every bounded convex symmetric domain in $\mathbf{R}^n$. Furthermore, if all brake orbits in this domain are nondegenerate, then there are at least n geometrically distinct brake orbits in it. In the symmetric case, we give a positive answer to the Seifert conjecture of 1948 under a generic condition.

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Keywords: Brake orbit; Maslov-type index; Bott-type iteration formula; Convex symmetric domain

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1. Introduction

Our aim of this paper is twofold. We first establish an iteration theory of the Maslov-type index associated with a Lagrangian subspace of $(\mathbb{R}^{2n}, \omega_0)$ for symplectic paths starting from identity. The Bott-type iteration formulas and some abstract precise iteration formulas are obtained here. Then as the application of this theory, we consider the brake orbit problem on a fixed energy hypersurface of the autonomous Hamiltonian systems. The multiplicity results are obtained in this paper.

1.1. Main results for the brake orbit problem

Let $V \in C^2(\mathbf{R}^n, \mathbf{R})$ and $h > 0$ such that $\Omega \equiv \{q \in \mathbf{R}^n \mid V(q) < h\}$ is nonempty, bounded, open and connected. Consider the following fixed energy problem of the second order autonomous Hamiltonian system

$$\ddot{q}(t) + V'\big(q(t)\big) = 0, \quad \text{for } q(t) \in \Omega, \tag{1.1}$$

$$\frac{1}{2}\big|\dot{q}(t)\big|^2 + V\big(q(t)\big) = h, \quad \forall t \in \mathbf{R}, \tag{1.2}$$

$$\dot{q}(0) = \dot{q}\left(\frac{\tau}{2}\right) = 0, \tag{1.3}$$

$$q\left(\frac{\tau}{2} + t\right) = q\left(\frac{\tau}{2} - t\right), \qquad q(t+\tau) = q(t), \quad \forall t \in \mathbf{R}. \tag{1.4}$$

A solution (τ, q) of (1.1)–(1.4) is called a *brake orbit* in Ω. We call two brake orbits q_1 and $q_2 : \mathbf{R} \to \mathbf{R}^n$ *geometrically distinct* if $q_1(\mathbf{R}) \neq q_2(\mathbf{R})$.

We denote by $\mathcal{O}(\Omega)$ and $\tilde{\mathcal{O}}(\Omega)$ the sets of all brake orbits and geometrically distinct brake orbits in Ω respectively.

Let $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and $N = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ with I being the identity in $\mathbf{R}^n$. Suppose that $H \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$ satisfying

$$H(Nx) = H(x), \quad \forall x \in \mathbf{R}^{2n}. \tag{1.5}$$

We consider the following fixed energy problem

$$\dot{x}(t) = JH'\big(x(t)\big), \tag{1.6}$$

$$H\big(x(t)\big) = h, \tag{1.7}$$

$$x(-t) = Nx(t), \tag{1.8}$$

$$x(\tau + t) = x(t), \quad \forall t \in \mathbf{R}. \tag{1.9}$$

A solution (τ, x) of (1.6)–(1.9) is also called a *brake orbit* on $\Sigma := \{y \in \mathbf{R}^{2n} \mid H(y) = h\}$.

Remark 1.1. It is well known that via

$$H(p, q) = \frac{1}{2}|p|^2 + V(q), \tag{1.10}$$