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Sobolev regularization of a class of forward–backward parabolic equations

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Abstract

We address existence and asymptotic behaviour for large time of *Young measure solutions* of the Dirichlet initial–boundary value problem for the equation $u_t = \nabla \cdot [\varphi(\nabla u)]$, where the function φ need not satisfy monotonicity conditions. Under suitable growth conditions on φ, these solutions are obtained by a "vanishing viscosity" method from solutions of the corresponding problem for the regularized equation $u_t = \nabla \cdot [\varphi(\nabla u)] + \epsilon \Delta u_t$. The asymptotic behaviour as $t \to \infty$ of Young measure solutions of the original problem is studied by ω-limit set techniques, relying on the *tightness* of sequences of time translates of the limiting Young measure. When $N = 1$ this measure is characterized as a linear combination of Dirac measures with support on the branches of the graph of φ.

Keywords: Forward–backward parabolic equations; Pseudo-parabolic regularization; Bounded Radon measures; Entropy inequalities

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1. Introduction

In this paper we consider the initial–boundary value problem

$$(P)\qquad \begin{cases} u_t = \nabla \cdot \big[\varphi(\nabla u)\big] & \text{in } Q_T := \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}. \end{cases}$$

Here $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ if $N \geq 2$, $T \in (0,\infty]$ and the dot "$\cdot$" denotes the scalar product in $\mathbb{R}^N$.

If $N = 1$, on the function $\varphi : \mathbb{R} \to \mathbb{R}$ which appears in problem (P) we shall assume the following:

$$(A_1)\qquad \begin{cases} \text{for any } R > 0 \text{ there exists } L_R > 0 \text{ such that} \\ \big|\varphi(\xi_1) - \varphi(\xi_2)\big| \leq L_R|\xi_1 - \xi_2| \text{ for any } \xi_1, \xi_2 \in B_R, \end{cases}$$

where $B_R \equiv (-R, R) \subset \mathbb{R}$ $(R > 0)$;

$$(A_2)\qquad \begin{cases} \text{there exist } \xi_0 > 0,\ p \in (1,\infty) \text{ and } C_1 > 0 \text{ such that} \\ C_1|\xi|^{p-1} \leq \big|\varphi(\xi)\big| \text{ for any } |\xi| > \xi_0; \end{cases}$$

$$(A_3)\qquad \varphi(\xi)\xi \geq 0 \quad \text{for any } \xi \in \mathbb{R}.$$

If $N \geq 2$, concerning the map $\varphi : \mathbb{R}^N \to \mathbb{R}^N$, $\varphi \equiv (\varphi_1, \dots, \varphi_N)$, the following assumptions will be made

$$(H_1)\qquad \begin{cases} \text{there exists } L > 0 \text{ such that} \\ \big|\varphi(\xi_1) - \varphi(\xi_2)\big| \leq L|\xi_1 - \xi_2| \text{ for any } \xi_1, \xi_2 \in \mathbb{R}^N; \end{cases}$$

$$(H_2)\qquad \begin{cases} \text{there exist } \xi_0 > 0,\ p \in (1,2] \text{ and } C_0 > 0 \text{ such that} \\ \big|\varphi(\xi)\big| \leq C_0\big(1 + |\xi|^{p-1}\big) \text{ for any } |\xi| > \xi_0; \end{cases}$$

$$(H_3)\qquad \text{there exists } \Phi \in C^1\big(\mathbb{R}^N\big) \text{ such that } \varphi = \nabla\Phi;$$

$$(H_4)\qquad \begin{cases} \text{there exist } \xi_0 > 0,\ q \in (1,2] \text{ and } C_1 > 0 \text{ such that} \\ C_1|\xi|^q \leq \Phi(\xi) \text{ for any } |\xi| > \xi_0; \end{cases}$$

$$(H_5)\qquad \varphi(\xi) \cdot \xi \geq 0 \quad \text{for any } \xi \in \mathbb{R}^N.$$

Observe that for $N \geq 2$ we assume global Lipschitz continuity of φ, instead of local Lipschitz continuity as in the case $N = 1$; this implies the stronger restriction $p \in (1,2]$ (instead of $p \in (1,\infty)$ as for $N = 1$) on the allowed values of p. Observe also that by (H_2)–(H_4) there holds

$$C_1|\xi|^q \leq \Phi(\xi) \leq C_3|\xi|^p \quad \text{for any } |\xi| > \xi_0, \tag{1.1}$$

for some constant $C_3 > 0$. This implies the compatibility condition $q \leq p$. In the following we always choose $q = p$ (in this connection, see Remark 3.2 below). Concerning the initial data function u_0,