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Semilinear fractional elliptic equations involving measures

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Abstract

We study the existence of weak solutions to (E) $(-\Delta)^\alpha u + g(u) = \nu$ in a bounded regular domain Ω in $\mathbb{R}^N (N \geq 2)$ which vanish in $\mathbb{R}^N \setminus \Omega$, where $(-\Delta)^\alpha$ denotes the fractional Laplacian with $\alpha \in (0, 1)$, ν is a Radon measure and g is a nondecreasing function satisfying some extra hypotheses. When g satisfies a subcritical integrability condition, we prove the existence and uniqueness of weak solution for problem (E) for any measure. In the case where ν is a Dirac measure, we characterize the asymptotic behavior of the solution. When $g(r) = |r|^{k-1}r$ with k supercritical, we show that a condition of absolute continuity of the measure with respect to some Bessel capacity is a necessary and sufficient condition in order (E) to be solved.

MSC: 35R11; 35J61; 35R06

Keywords: Fractional Laplacian; Radon measure; Dirac measure; Green kernel; Bessel capacities

Contents

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain (i.e. a nonempty bounded and connected open set with a C^2 boundary), $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function and $\mathfrak{M}(\Omega, \rho^\beta)$ be the space of Radon measures in Ω satisfying $\int_\Omega \rho^\beta d|\nu| < +\infty$ with $\rho(x) = dist(x, \Omega^c)$, especially, $\mathfrak{M}(\Omega, \rho^0) = \mathfrak{M}^b(\Omega)$ be the set of bounded Radon measures. We are concerned with the existence of weak solutions to the semilinear fractional elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= \nu \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Omega^c, \end{aligned} \tag{1.1}$$

where $\alpha \in (0, 1)$, $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, \alpha]$. The fractional Laplacian $(-\Delta)^\alpha$ is defined by

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \to 0^+} (-\Delta)^\alpha_\epsilon u(x),$$

where for $\epsilon > 0$,

$$(-\Delta)^\alpha_\epsilon u(x) = -\int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\epsilon\big(|x - z|\big) dz \tag{1.2}$$

and

$$\chi_\epsilon(t) = \begin{cases} 0, & \text{if } t \in [0, \epsilon], \\ 1, & \text{if } t > \epsilon. \end{cases}$$

When $\alpha = 1$, the semilinear elliptic problem

$$\begin{aligned} -\Delta u + g(u) &= \nu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

has been extensively studied by numerous authors in the last 30 years. A fundamental contribution is due to Brezis [7], Bénilan and Brezis [2], where $\nu \in \mathfrak{M}^b(\Omega)$ and the function $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing, positive on $(0, +\infty)$ and satisfies the subcritical assumption: