Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 257 (2014) 1529–1566

Journal of Differential Equations

www.elsevier.com/locate/jde

Existence of entire solutions for a class of variable exponent elliptic equations

Patrizia Pucci [a], Qihu Zhang [b,*]

[a] *Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Via Vanvitelli 1, 06123 Perugia, Italy*
[b] *College of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, China*

Received 27 March 2014; revised 6 May 2014

Available online 2 June 2014

Dedicated to Prof. Xianling Fan on the occasion of his 70th birthday

Abstract

The paper deals with the existence of entire solutions for a quasilinear equation $(\mathcal{E}_\lambda)$ in $\mathbb{R}^N$, depending on a real parameter λ, which involves a general variable exponent elliptic operator $\mathbf{A}$ in divergence form and two main nonlinearities. The competing nonlinear terms combine each other. Under some conditions, we prove the existence of a critical value $\lambda_* > 0$ with the property that $(\mathcal{E}_\lambda)$ admits nontrivial nonnegative entire solutions if and only if $\lambda \geq \lambda_*$. Furthermore, under the further assumption that the potential $\mathscr{A}$ of $\mathbf{A}$ is uniform convex, we give the existence of a second independent nontrivial nonnegative entire solution of $(\mathcal{E}_\lambda)$, when $\lambda > \lambda_*$. Our results extend the previous work of Autuori and Pucci (2013) [6] from the case of constant exponents p, q and r to the case of variable exponents. More interesting, we weaken the condition $\max\{2, p\} < q < \min\{r, p^*\}$ to the simple request that $1 \ll q \ll r$. Furthermore, we extend the previous work of Alama and Tarantello (1996) [2] from Dirichlet Laplacian problems in bounded domains of $\mathbb{R}^N$ to the case of a general variable exponent differential equation in the entire $\mathbb{R}^N$, and also remove the assumption $q > 2$. Hence the results of this paper are new even in the canonical case $p(\cdot) \equiv 2$.

MSC: primary 35J62, 35J70; secondary 35J20

Keywords: $p(x)$-Laplacian; Integral functionals; Variable exponent Sobolev spaces; Critical points

* Corresponding author.
E-mail addresses: patrizia.pucci@unipg.it (P. Pucci), zhangqihu@yahoo.com, zhangqh1999@yahoo.com.cn (Q. Zhang).

http://dx.doi.org/10.1016/j.jde.2014.05.023
0022-0396/

1. Introduction

In this paper we study the one parameter variable exponent elliptic equation

$$-\operatorname{div}\mathbf{A}(x,\nabla u)+a(x)|u|^{p(x)-2}u=\lambda w(x)|u|^{q(x)-2}u-h(x)|u|^{r(x)-2}u \qquad (\mathcal{E}_\lambda)$$

in $\mathbb{R}^N$, where $\lambda\in\mathbb{R}$ and $\mathbf{A}:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ *admits a potential* $\mathscr{A}$*, with respect to its second variable* ξ*, and satisfies the following assumption, required throughout the paper.*

($\mathcal{A}_1$) *The potential* $\mathscr{A}=\mathscr{A}(x,\xi)$ *is a continuous function in* $\mathbb{R}^N\times\mathbb{R}^N$*, with continuous derivative with respect to* ξ, $\mathbf{A}=\partial_\xi\mathscr{A}(x,\xi)$*, and verifies:*
- *(i)* $\mathscr{A}(x,0)=0$ *and* $\mathscr{A}(x,\xi)=\mathscr{A}(x,-\xi)$ *for all* $(x,\xi)\in\mathbb{R}^N\times\mathbb{R}^N$*;*
- *(ii)* $\mathscr{A}(x,\cdot)$ *is strictly convex in* $\mathbb{R}^N$ *for all* $x\in\mathbb{R}^N$*;*
- *(iii) There exist constants* $C_1, C_2>0$ *and an exponent* $p\in\mathcal{P}^{\log}(\mathbb{R}^N)$ *such that*

$$C_1|\xi|^{p(x)}\le\mathbf{A}(x,\xi)\cdot\xi,\qquad \big|\mathbf{A}(x,\xi)\big|\le C_2|\xi|^{p(x)-1} \qquad (1)$$

for all $(x,\xi)\in\mathbb{R}^N\times\mathbb{R}^N$*, where* $1\ll p\ll N$*, and* $p\in\mathcal{P}^{\log}(\mathbb{R}^N)$ *means that* p *is globally* log*-Hölder continuous in* $\mathbb{R}^N$*, i.e. there exists a* p_∞ *satisfying*

$$\big|p(x)-p(y)\big|\le C_3/\ln\big(e+1/|x-y|\big),$$
$$\big|p(x)-p_\infty\big|\le C_4/\ln\big(e+|x|\big)$$

for all $x, y\in\mathbb{R}^N$.

A typical example is $\mathbf{A}(x,\xi)=|\xi|^{p(x)-2}\xi$, so that along any $\varphi\in C_0^\infty(\mathbb{R}^N)$

$$-\operatorname{div}\mathbf{A}(x,\nabla\varphi)=-\operatorname{div}\big(|\nabla\varphi|^{p(x)-2}\nabla\varphi\big)=-\Delta_{p(x)}\varphi,$$

which is called the $p(x)$-Laplacian and satisfies ($\mathcal{A}_1$) provided that $p\in\mathcal{P}^{\log}(\mathbb{R}^N)$ and $1<p^-\le p^+<N$, where

$$p^+=\sup_{x\in\mathbb{R}^N}p(x),\qquad p^-=\inf_{x\in\mathbb{R}^N}p(x).$$

We write $p_1\ll p_2$ if

$$\inf_{x\in\mathbb{R}^N}\big[p_2(x)-p_1(x)\big]>0,$$

where p_1 and p_2 are two continuous exponents. Furthermore, *in the entire paper, we assume*

($\mathcal{H}_1$) (i) $a\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, *and for some constant* $c_1\in(0,1]$ *the coefficient* a *satisfies*

$$a(x)\ge c_1\big(1+|x|\big)^{-p(x)}\quad \textit{for all } x\in\mathbb{R}^N; \qquad (2)$$