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Hamiltonian linear type centers of linear plus cubic homogeneous polynomial vector fields

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Abstract

We provide normal forms and the global phase portraits in the Poincaré disk for all the Hamiltonian linear type centers of linear plus cubic homogeneous planar polynomial vector fields.

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Keywords: Hamiltonian linear type center; Cubic polynomial; Vector fields; Phase portrait

1. Introduction and statement of the main results

In the qualitative theory of real planar polynomial differential systems two of the main problems are the determination of limit cycles and the center-focus problem, i.e. to distinguish when a singular point is either a focus or a center. The notion of *center* goes back to Poincaré in [20]. He defined a center for a vector field on the real plane as a singular point having a neighborhood filled of periodic orbits with the exception of the singular point.

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The classification of the centers of quadratic differential systems started with the works of Dulac [10], Kapteyn [14,15] and Bautin [2]. The bifurcation diagrams of all quadratic differential systems possessing a center were done by Schlomiuk [22] and Żołądek [25]. There are many partial results for the centers of polynomial differential systems of degree larger than 2. We must mention that Malkin [17], and Vulpe and Sibirsky [24] characterized the centers of the polynomial differential systems with linear and homogeneous nonlinearities which have a pair of eigenvalues of the form $\pm\omega i$ with $\omega \neq 0$, while we characterize for the same family of polynomial differential systems the nilpotent centers, which in particular have two zero eigenvalues. We note that Malkin, and Vulpe and Sibirsky did not provide the global phase portraits of the centers with eigenvalues $\pm\omega i$. In [9] we characterize the global phase portraits of the systems studied by them but having nilpotent centers.

For polynomial differential systems of the form linear with homogeneous nonlinearities of degree greater than 3 the centers at the origin are not characterized, but there are partial results for degrees 4 and 5 for the centers with eigenvalues $\pm\omega i$ with $\omega \neq 0$, see for instance Chavarriga and Giné [3,4]. On the other hand there is a long way to go for obtaining the complete classification of the centers for all polynomial differential systems of degree 3. Some interesting results on some subclasses of cubic systems are those of Rousseau and Schlomiuk [23], and the ones of Żołądek [26,27].

We say that a singular point is *non-elementary* if both of the eigenvalues of the linear part of the vector field at that point are zero, and *elementary* otherwise. A non-elementary singular point is called *degenerate* if the linear part is identically zero, otherwise it is called *nilpotent*. In accordance with these definitions, if an analytic system has a center, then after an affine change of variables and a rescaling of the time variable, it can be written in one of the following three forms:

$$\dot{x} = -y + P(x, y), \qquad \dot{y} = x + Q(x, y),$$

called a *linear type center*;

$$\dot{x} = y + P(x, y), \qquad \dot{y} = Q(x, y),$$

called a *nilpotent center*;

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y),$$

called a *degenerate center*, where $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin.

There is an algorithm for the characterization of linear type centers due to Poincaré [21] and Lyapunov [16], see also Chazy [6] and Moussu [18]. An algorithm for the characterization of the nilpotent and some class of degenerate centers is done in the works of Chavarriga et al. [5], Giacomini et al. [13], and Cima and Llibre [8].

In this work we provide the global phase portraits of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center at the origin. To do this we will use the Poincaré compactification of polynomial vector fields, see