Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 257 (2014) 1662–1688
Journal of Differential Equations
www.elsevier.com/locate/jde

Bifurcations and complex dynamics of an SIR model with the impact of the number of hospital beds ☆

Chunhua Shan, Huaiping Zhu *

Department of Mathematics and Statistics, York University, Toronto, ON, M3J 1P3, Canada

Received 14 July 2012; revised 9 April 2014
Available online 3 June 2014

Abstract

In this paper we establish an SIR model with a standard incidence rate and a nonlinear recovery rate, formulated to consider the impact of available resource of the public health system especially the number of hospital beds. For the three dimensional model with total population regulated by both demographics and diseases incidence, we prove that the model can undergo backward bifurcation, saddle-node bifurcation, Hopf bifurcation and cusp type of Bogdanov–Takens bifurcation of codimension 3. We present the bifurcation diagram near the cusp type of Bogdanov–Takens bifurcation point of codimension 3 and give epidemiological interpretation of the complex dynamical behaviors of endemic due to the variation of the number of hospital beds. This study suggests that maintaining enough number of hospital beds is crucial for the control of the infectious diseases.

Keywords: SIR model; Nonlinear recovery rate; Number of hospital beds; Dynamics; Backward bifurcation; Saddle-node bifurcation; Hopf bifurcations; Bogdanov–Takens bifurcations

1. Introduction

There is a long and distinguished history of mathematical modeling in epidemiology since Bernoulli developed the first dynamic model of smallpox spread in 1760 [3]. The study

☆ This research was supported by NSERC and ERA, an Early Researcher Award of Ministry of Research and Innovation of Ontario, Canada.

* Corresponding author.
E-mail address: huaiping@mathstat.yorku.ca (H. Zhu).

http://dx.doi.org/10.1016/j.jde.2014.05.030

of epidemiological models reveals the underlying mechanisms that influence the transmission and control of infectious diseases. Among numerous epidemiological models, the SIR type of compartmental models and Ross–MacDonald models have played a critical role in the development of modern mathematical epidemiology, and mathematical modeling approach has become an important tool in understanding the dynamics and mechanisms of disease transmission for the purpose of control and prevention of infectious diseases [1,3,6].

In modeling of infectious diseases, there are several factors that substantially affect the dynamical behavior of the models such as the demographics of involving populations, incidence rate and recovery rate. In the classic SIR models, the simple mass action incidence rate or the standard incidence rate $\frac{\beta S(t)I(t)}{N(t)}$ and linear recovery rate $\mu I(t)$ are used, where $S(t)$, $I(t)$ and $R(t)$ are the numbers of susceptible, infectious and recovered individuals at time t, respectively. $N(t) = S(t) + I(t) + R(t)$ is the number of the total populations. The constant β is the average number of adequate contacts per unit time with infectious individuals, and the constant μ is the per capita recovery rate of infectious individuals. For classic epidemiological models with different demographics, typically they do not have bistability and periodicity, and the dynamics are almost completely characterized by the so called basic reproduction numbers $\mathbb{R}_0$. The disease will be eliminated if $\mathbb{R}_0 < 1$, otherwise the disease will persist [3,6,10,14].

In practice, multiple peaks or periodic oscillations are observed during the transmission of many infectious disease. Lots of studies have shown that the nonlinear incidence rate plays a crucial role in producing the rich dynamics of epidemic models including periodic oscillations [4,5,9,10,12–16] etc. The saturated incidence rate $g(I)S$ is introduced into the epidemic model by Capasso and Serio [4]. The nonlinear incidence rate $\kappa I^p S^q$ $(\kappa, p, q > 0)$ is investigated by Liu et al. [12,13]. The effect of behavioral changes has been incorporated by Liu using the nonlinear incidence rate $\frac{\kappa I^l S}{1+\alpha I^h}$ with $\kappa, l, \alpha, h > 0$, and studied by Liu et al. [12,13] and Ruan and their colleagues [15,16]. Generalized form of nonlinear incidence rate is considered by Derrick and van den Driessche [5].

The SIR and SIRS models always assume that the total population size remains or asymptotically approaches a constant. This assumption is reasonable when the disease spreads quickly and dies out within a short time or the disease rarely causes deaths, which can help to reduce the model into a planar system to make the mathematical analysis easier as many tools can be used for the planar system. However, it is not reasonable when the natural birth and death rate are not balanced or when the disease-induced deaths are significant such as plague, measles, scarlet fever, diphtheria, tuberculosis, smallpox, malaria etc. In this study, we suppose the diseased-induced death rate $\nu > 0$ so that the human populations will be regulated by both the demographics and infectious diseases.

In this paper we consider the following SIR model with the standard incidence rate [14]

$$\begin{cases} \dfrac{dS}{dt} = A - dS - \dfrac{\beta SI}{S+I+R}, \\ \dfrac{dI}{dt} = -(d+\nu)I - \mu I + \dfrac{\beta SI}{S+I+R}, \\ \dfrac{dR}{dt} = \mu I - dR, \end{cases} \tag{1.1}$$

where $A > 0$ is the recruitment rate of susceptible population; $d > 0$ is the per capita natural death rate of the population; $\nu > 0$ is the per capita disease-induced death rate; $\mu > 0$ is the per capita recovery rate of infectious individuals.