Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 257 (2014) 351–373
Journal of Differential Equations
www.elsevier.com/locate/jde

Non-autonomous second order Hamiltonian systems

John Pipan, Martin Schechter *

Department of Mathematics, University of California, Irvine, CA 92697-3875, USA

Received 1 July 2013; revised 26 March 2014
Available online 24 April 2014

Abstract

We study the existence of periodic solutions for a second order non-autonomous dynamical system containing variable kinetic energy terms. Our assumptions balance the interaction between the kinetic energy and the potential energy with neither one dominating the other. We study sublinear problems and the existence of non-constant solutions.

MSC: 35J20; 35J25; 35J60; 35Q55; 35J65; 47J30; 49B27; 49J40; 58E05

Keywords: Critical points; Linking; Dynamical systems; Periodic solutions

1. Introduction

We consider the following problem. One wishes to solve

$$-\ddot{x}(t) = B(t)x(t) + \nabla_x V\big(t, x(t)\big), \tag{1}$$

where

$$x(t) = \big(x_1(t), \dots, x_n(t)\big) \tag{2}$$

is a map from $I = [0, T]$ to $\mathbb{R}^n$ such that each component $x_j(t)$ is a periodic function in H^1 with period T, and the function $V(t, x) = V(t, x_1, \dots, x_n)$ is continuous from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ with

$$\nabla_x V(t, x) = (\partial V/\partial x_1, \dots, \partial V/\partial x_n) \in C\big(\mathbb{R}^{n+1}, \mathbb{R}^n\big). \tag{3}$$

* Corresponding author.

http://dx.doi.org/10.1016/j.jde.2014.03.016
0022-0396/

For each $x \in \mathbb{R}^n$, the function $V(t, x)$ is periodic in t with period T. The elements of the symmetric matrix $B(t)$ are to be real-valued functions $b_{jk}(t) = b_{kj}(t)$, and each function is to be periodic with period T. We will consider each function to be defined on the interval I.

We shall study this problem under the following assumptions. Our assumption on $B(t)$ is:

$(B1)$ Each component of $B(t)$ is an integrable function on I, i.e., for each j and k, $b_{jk}(t) \in L^1(I)$.

This assumption implies that there is a linear operator $\mathcal{D}$ depending on $B(t)$ having spectrum consisting only of isolated eigenvalues of finite multiplicities tending to ∞. ($\mathcal{D}$ is defined in the next section.)

Concerning the potential $V(t, x)$ we assume:

$(V1)$ There exist functions $W_1, W_2 \in L^1(I)$ and consecutive eigenvalues λ_l, λ_{l+1} of $\mathcal{D}$ such that for any $t \in I$ and any $x \in \mathbb{R}^n$,

$$\lambda_l|x|^2 - W_1(t) \leqslant 2V(t, x) \leqslant \lambda_{l+1}|x|^2 + W_2(t). \tag{4}$$

$(V2)$ There exists a function $W_0(t) \in L^1(I)$ such that for any $t \in I$ and any $x \in \mathbb{R}^n$,

$$H(t, x) = 2V(t, x) - x \cdot \nabla_x V(t, x) \geqslant -W_0(t).$$

$(V3)$ $H(t, x) \to \infty$ uniformly in t as $|x| \to \infty$.

Assumptions $(V2)$ and $(V3)$ can be replaced by:

$(V2')$ There exists a function $W_0(t) \in L^1(I)$ such that for any $t \in I$ and any $x \in \mathbb{R}^n$,

$$H(t, x) \leqslant W_0(t),$$

and

$(V3')$ $H(t, x) \to -\infty$ uniformly in t as $|x| \to \infty$.

Then we have

Theorem 1.1. *If the functions $B(t)$ and $V(t, x)$ satisfy assumptions $(B1)$, $(V1)$, $(V2)$ and $(V3)$, then there exists a T-periodic weak solution to (1) whose weak second derivative is an element of $L^1(I)$. If the function $V(t, x)$ satisfies $\nabla_x V(t, \mathbf{0}) \neq \mathbf{0}$, the solution of (1) is not trivial. The conclusions are also valid if we replace assumptions $(V2)$ and $(V3)$ with $(V2')$ and $(V3')$.*

In all of the previous results dealing with the full system (1), the hypotheses cause either the linear terms (kinetic energy) to dominate the nonlinear terms (potential energy) or vice versa. In either case, the subordinate terms become perturbations of the dominant terms. In the present paper each accommodates the other; neither is dominant.