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Existence and conditional energetic stability of three-dimensional fully localised solitary gravity-capillary water waves

B. Buffoni [a], M.D. Groves [b,c,*], S.M. Sun [d], E. Wahlén [e]

[a] *Section de mathématiques (MATHAA), Station 8, École polytechnique fédérale, 1015 Lausanne, Switzerland*
[b] *FR 6.1 – Mathematik, Universität des Saarlandes, Postfach 151150, 66041 Saarbrücken, Germany*
[c] *Department of Mathematical Sciences, Loughborough University, Loughborough, Leics, LE11 3TU, UK*
[d] *Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA*
[e] *Centre for Mathematical Sciences, Lund University, P.O. Box 118, 22100 Lund, Sweden*

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ABSTRACT

In this paper we show that the hydrodynamic problem for three-dimensional water waves with strong surface-tension effects admits a *fully localised solitary wave* which decays to the undisturbed state of the water in every horizontal direction. The proof is based upon the classical variational principle that a solitary wave of this type is a critical point of the energy, which is given in dimensionless coordinates by

$$\mathcal{E}(\eta,\phi)=\int_{\mathbb{R}^2}\left\{\frac{1}{2}\int_0^{1+\eta}\left(\phi_x^2+\phi_y^2+\phi_z^2\right)\mathrm{d}y+\frac{1}{2}\eta^2+\beta\left[\sqrt{1+\eta_x^2+\eta_z^2}-1\right]\right\}\mathrm{d}x\,\mathrm{d}z,$$

subject to the constraint that the momentum

$$\mathcal{I}(\eta,\phi)=\int_{\mathbb{R}^2}\eta_x\phi|_{y=1+\eta}\,\mathrm{d}x\,\mathrm{d}z$$

is fixed; here $\{(x,y,z)\colon\ x,z\in\mathbb{R},\ \ y\in(0,1+\eta(x,z))\}$ is the fluid domain, ϕ is the velocity potential and $\beta>1/3$ is the Bond number.

* Corresponding author at: FR 6.1 – Mathematik, Universität des Saarlandes, Postfach 151150, 66041 Saarbrücken, Germany.
E-mail address: groves@math.uni-sb.de (M.D. Groves).

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These functionals are studied locally for η in a neighbourhood of the origin in $H^3(\mathbb{R}^2)$.
We prove the existence of a minimiser of $\mathcal{E}$ subject to the constraint $\mathcal{I} = 2\mu$, where $0 < \mu \ll 1$. The existence of a small-amplitude solitary wave is thus assured, and since $\mathcal{E}$ and $\mathcal{I}$ are both conserved quantities a standard argument may be used to establish the stability of the set D_μ of minimisers as a whole. 'Stability' is however understood in a qualified sense due to the lack of a global well-posedness theory for three-dimensional water waves. We show that solutions to the evolutionary problem starting near D_μ remain close to D_μ in a suitably defined energy space over their interval of existence; they may however explode in finite time due to higher-order derivatives becoming unbounded.

1. Introduction

1.1. The hydrodynamic problem

The classical *three-dimensional gravity-capillary water wave problem* concerns the irrotational flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. The fluid motion is described by the Euler equations in a domain bounded below by a rigid horizontal bottom $\{y = 0\}$ and above by a free surface $\{y = h + \eta(x, z, t)\}$, where h denotes the depth of the water in its undisturbed state and the function η depends upon the two horizontal spatial directions x, z and time t. In terms of an Eulerian velocity potential ϕ, the mathematical problem is to solve Laplace's equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad 0 < y < h + \eta,$$

with boundary conditions

$$\begin{aligned} &\phi_y = 0, && y = 0,\\ &\eta_t = \phi_y - \eta_x\phi_x - \eta_z\phi_z, && y = h + \eta,\\ &\phi_t = -\frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) - g\eta \\ &\qquad + \sigma\left[\frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_x + \sigma\left[\frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_z, && y = h + \eta, \end{aligned}$$

in which g is the acceleration due to gravity and $\sigma > 0$ is the coefficient of surface tension (see, for example, Stoker [30, §§1, 2.1]). Introducing the dimensionless variables

$$\left(x', y', z'\right) = \frac{1}{h}(x, y, z), \qquad t' = \left(\frac{g}{h}\right)^{1/2},$$

$$\eta'\left(x', z', t'\right) = \frac{1}{h}\eta(x, z, t), \qquad \phi'\left(x', y', z', t'\right) = \frac{1}{(gh)^{3/2}}\phi(x, y, z, t),$$

one obtains the equations

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad 0 < y < 1 + \eta, \tag{1}$$