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Generalized stochastic evolution equations

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Abstract

An approach to generalized stochastic evolution equations is presented which is based on a generalized Ito formula. This allows the consideration of interesting examples which are stochastic generalizations of evolution equations of mixed type or second order in time hyperbolic equations. It includes more standard material involving a Gelfand triple of spaces as a special case. Several examples are given which illustrate the use of the abstract theory presented.
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1. Introduction

This paper deals with generalized stochastic evolution equations of the form

$$Bu(t,\omega) - Bu_0(\omega) + \int_0^t A\big(s, u(t,\omega), \omega\big)ds = \int_0^t f(s,\omega)ds + B\int_0^t \Phi dW,$$

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the term in the end being a stochastic integral and B an operator which could be degenerate. We prove the existence and uniqueness of a global strong solution, by which we mean one which is adapted to the filtration coming from the given (arbitrary but fixed) Brownian motion W such that the given initial condition holds a.e. and the integral equation holds for a.e. ω and all t in any fixed interval $[0, T]$. The stochastic integral is the Ito integral. The main assumptions we make on A are monotonicity, hemicontinuity, and coercivity. So stochastic perturbation of equations like

$$b(x)u_t(t,x) = \nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right)$$

and

$$u_{tt} = -u_{xxxx} + f$$

could be studied. This will be made clear by our examples.

The theory for the case where $B = I$, the identity map, is well established. A well known work is [6], which generalizes earlier work in [4]. The main idea is to consider finite approximate problems, make energy estimates using the Ito formula and other standard techniques in evolution equations, take weak limits, and then identify using monotonicity conditions. Also refer to [7] and references therein for applications.

This work is a nontrivial generalization of [6]. The novelty is in allowing B to be an operator which could vanish somewhere, or be noninvertible, or have other interesting features. This will generate many substantial difficult issues. For instance, the measurability into different spaces, the use of a generalized Ito formula, and the identification of weak limits and so on. The essential difference is that standard energy estimates like those in [6] which are based on taking expectation do not provide enough information due to the possible degeneracy of the operator B. We need a stronger uniform energy estimate which holds both pointwise and in expectation. We will emphasize the differences between this work and [6] by remarks in several places. The key energy estimate is Lemma 15, which utilizes theorems like Prokhorov's theorem, Skorokhod's theorem, and Kolmogorov extension theorem.

As a starting point, we assume in our general theorem, Theorem 6, monotonicity, hemicontinuity, and coercivity so that path uniqueness is available. We show by examples how stochastic perturbation of PDEs, which are of mixed type or second order in time, could be studied under our generalized setting. Later we will use the theory developed her to consider stochastic inclusions.

We organize the paper in the following manner. In Section 2, we state the precise assumptions, and list some useful lemmas. As a basis, we give the generalized Ito formula of [5]. In Section 3, we consider the approximate problem under two sets of assumptions. In Section 4, we consider the general case and prove the existence and uniqueness of strong solution of the generalized stochastic evolution equation. Then in Section 5, we give some examples.

2. Hypothesis and preliminary results

2.1. Assumptions and some useful lemmas

In this paper, V is a separable Banach space, W is a separable Hilbert space. Suppose V is dense in W. So

$$V \subseteq W, \qquad W' \subseteq V',$$