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Concentration behavior of standing waves for almost mass critical nonlinear Schrödinger equations

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Abstract

We study the following nonlinear Schrödinger equation

$$iu_t = -\Delta u + V(x)u - a|u|^q u, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^2,$$

where $a > 0$, $q \in (0, 2)$, and $V(x)$ is some type of trapping potential. For any fixed $a > a^* := \|Q\|_2^2$, where Q is the unique (up to translations) positive radial solution of $\Delta u - u + u^3 = 0$ in $\mathbb{R}^2$, by directly using constrained variational method and energy estimates we present a detailed analysis of the concentration and symmetry breaking of standing waves for the above equation as $q \nearrow 2$.

MSC: 35J20; 35J60

Keywords: Constrained variational method; Energy estimates; Concentration; Standing waves; Nonlinear Schrödinger equation; Symmetry breaking

1. Introduction

In this paper, we study the concentration and symmetry breaking of standing waves for the following nonlinear Schrödinger equation (NLS) with a trapping potential and an attractive nonlinearity

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$$iu_t = -\Delta u + V(x)u - a|u|^q u, \quad (t,x) \in \mathbb{R}^1 \times \mathbb{R}^2, \tag{1.1}$$

where $a > 0$, $0 < q < 2$, and $V(x)$ is a trapping potential. Eq. (1.1) with $q = 2$ arises in Bose–Einstein condensates (BEC) as well as nonlinear optics, which has been studied widely in recent years, see for examples, [6,9,18,23,28]. In fact, when $q = 2$ the above equation (1.1) is the so-called mass critical NLS in $\mathbb{R}^2$, so $q = 2$ is usually called a mass critical exponent for (1.1). This paper is focused on the case where q approaches 2 from the left ($q \nearrow 2$, in short), which is what we mean by the almost mass critical NLS.

For (1.1), the standing waves are the solutions of (1.1) with the form: $u(t,x) = e^{i\omega t}\varphi_\omega(x)$, which implies that $\varphi_\omega(x)$ satisfies the following elliptic partial differential equation

$$-\Delta u + (V + \omega)u - a|u|^q u = 0 \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

When $q = 2$, (1.2) is also called the time-independent Gross–Pitaevskii (GP) equation of Bose–Einstein condensates, where ω represents the chemical potential, V is an external potential, and a is a coupling constant related to the number of bosons in a quantum system. Here $a > 0$ (*resp*. < 0) means that the BEC is attractive (*resp*. repulsive). In this paper, we consider only the attractive case, i.e., $a > 0$. It is well known that a minimizer of the following Gross–Pitaevskii (GP) energy functional

$$E_q(u) := \int\limits_{\mathbb{R}^2} \left(\left|\nabla u(x)\right|^2 + V(x)\left|u(x)\right|^2\right) dx - \frac{2a}{q+2} \int\limits_{\mathbb{R}^2} \left|u(x)\right|^{q+2} dx \tag{1.3}$$

under the following constraint

$$\int\limits_{\mathbb{R}^2} u^2\, dx = 1 \tag{1.4}$$

solves (1.2) for some Lagrange multiplier $\omega \in \mathbb{R}$. Based on these observations, to seek the standing waves of (1.1) we need only to get solutions of (1.2), and this can be done by solving the following constrained minimization problem associated with GP energy (1.3)

$$d_a(q) := \inf_{\{u \in \mathcal{H},\ \int_{\mathbb{R}^2} u^2\, dx = 1\}} E_q(u), \tag{1.5}$$

where $\mathcal{H}$ is defined by

$$\mathcal{H} := \left\{ u \in H^1\left(\mathbb{R}^2\right): \int\limits_{\mathbb{R}^2} V(x)\left|u(x)\right|^2 dx < \infty \right\}. \tag{1.6}$$

Here $V(x) : \mathbb{R}^2 \to \mathbb{R}^+$ is locally bounded and satisfies $V(x) \to \infty$ as $|x| \to \infty$. Without loss of generality, by adding a suitable constant we may assume that

$$\inf_{x \in \mathbb{R}^2} V(x) = 0,$$