Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 256 (2014) 2115–2151

Journal of Differential Equations

www.elsevier.com/locate/jde

Well-posedness and spectral properties of heat and wave equations with non-local conditions

Delio Mugnolo [a,*], Serge Nicaise [b]

[a] *Institut für Analysis, Universität Ulm, 89069 Ulm, Germany*

[b] *Université de Valenciennes et du Hainaut Cambrésis, LAMAV, FR CNRS 2956, ISTV, F-59313 Valenciennes Cedex 9, France*

Received 10 May 2012; revised 14 August 2013

Available online 17 January 2014

Abstract

We consider the one-dimensional heat and wave equations but – instead of boundary conditions – we impose on the solution certain non-local, integral constraints. An appropriate Hilbert setting leads to an integration-by-parts formula in Sobolev spaces of negative order and eventually allows us to use semigroup theory leading to analytic well-posedness, hence sharpening regularity results previously obtained by other authors. In doing so we introduce a parametrization of such integral conditions that includes known cases but also shows the connection with more usual boundary conditions, like periodic ones. In the self-adjoint case, we even obtain eigenvalue asymptotics of so-called Weyl's type.

MSC: 47D06; 35J20; 34B10

Keywords: Non-local conditions; Analytic semigroups; Quadratic forms; Weyl asymptotics

1. Introduction

Several problems in the applied sciences are modeled by partial differential equations on domains that, as such, require the modeler to impose some assumption on the sought-after solution in order to obtain well-posedness in some function space. Typical are, of course, boundary conditions like Dirichlet or Neumann ones. However, in many physical problems certain different constraints are natural: for example, all equations that are derived from conservation laws – like

* Corresponding author.
E-mail addresses: delio.mugnolo@uni-ulm.de (D. Mugnolo), Serge.Nicaise@univ-valenciennes.fr (S. Nicaise).

0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2013.12.016

the Cahn–Hilliard equation or the Navier–Stokes equation – admit the conservation of a certain physical quantity (e.g. mass, barycenter, energy, or momentum) and it is natural to wonder whether already these *minimal* constraints suffice to obtain well-posedness, at least for a special choice of the initial conditions. While this idea seems to be widely applicable, for simplicity we restrict this paper to the heat and the wave equations on an interval. Instead of usual boundary conditions on the solution u, we investigate the role of non-local, integral conditions like

$$\int_0^1 u(t,x)\,dx = 0, \quad t \geqslant 0. \tag{1.1}$$

This amounts to imposing that the moment of order 0 (corresponding e.g. to the total mass, in the case of a diffusion equation for which u denotes the relative density of a mixture) vanishes identically. A heat equation complemented by the above condition has been introduced by J.R. Cannon in [10], where well-posedness was investigated by methods based on abstract Volterra equations. While Cannon's work has received much attention by numerical analysts, it has gone largely overlooked by the PDE community, with some notable exceptions (cf. [18,32,31] and the references therein to earlier Soviet literature). The fact that (1.1) only eliminates one degree of freedom still forced Cannon and later investigators to impose a local (say, Dirichlet) condition at one of the endpoints.

More recently, it has been observed that the local condition at 0 or 1 may be dropped and replaced by another condition on the moment of order 1, like

$$\int_0^1 (1-x)u(t,x)\,dx = 0, \quad t \geqslant 0. \tag{1.2}$$

Wave and heat equations with (generalizations of) conditions (1.1) and (1.2) have been intensively studied by A. Bouziani and L.S. Pul'kina in a long series of papers that seems to begin with [28] and [7]. In [5], a condition on the moment of order 2 is discussed. In fact, over the last 20 years Bouziani, Pul'kina and their coauthors have discussed a manifold of hyperbolic, parabolic and pseudoparabolic equations with such conditions, mostly by numerical methods. Among others, in [6,14,17] several weaker well-posedness results for related parabolic, hyperbolic or pseudoparabolic equations have been obtained by different methods. An extensive list of further papers treating these or similar conditions can be found in the introduction of [15]. A few tentative extensions of the above conditions for heat or wave equations on higher dimensional domains have been proposed in the literature, cf. [23,29]. We are not aware of earlier investigations about the possibility of replacing a condition on the moment of order one by a condition on some moment of higher order. In the companion papers [25,24] we have further developed our techniques in order to address these issues.

The main goal of this article is to provide an abstract framework – as general as possible – for studying the one-dimensional heat or wave equation with integral conditions by means of semigroup theory. It turns out that, for the spatial operator we are considering, the associated diffusion equation is the gradient flow of a very simple functional – up to lower order terms, it is simply the L^2-norm – with respect to some H^{-1}-type inner product. This is not surprising: for example, it is well-known that the gradient flow associated to the L^p-norm with respect to the $H^{-1}(0,1)$-inner product is the porous medium equation with Dirichlet boundary conditions.