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ScienceDirect
J. Differential Equations 256 (2014) 2194–2245
Journal of Differential Equations
www.elsevier.com/locate/jde

On a class of first order Hamilton–Jacobi equations in metric spaces

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Received 5 June 2013; revised 20 December 2013
Available online 17 January 2014

Abstract

We establish well-posedness of a class of first order Hamilton–Jacobi equation in geodesic metric spaces. The result is then applied to solve a Hamilton–Jacobi equation in the Wasserstein space of probability measures, which arises from the variational formulation of a compressible Euler equation.
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1. Introduction

Let (X, d) be a complete metric space and a geodesic space. We are interested in a class of minimization problem which includes in particular the following:

$$\int_0^T \left(\frac{1}{2}|x'|^2 - V(x)\right) dr$$

where $V : \mathsf{X} \mapsto \mathbb{R}$ is uniformly continuous and bounded from above, $x = x(t) \in AC([0, T]; \mathsf{X})$ is an absolutely continuous path in X, and

$$|x'|(t) := \lim_{s \to t} \frac{\mathsf{d}(x(s), x(t))}{|s - t|}$$

denotes its metric derivative. See Chapter 1, Ambrosio, Gigli and Savaré [2] for definitions and properties of absolutely continuous curves in metric spaces. Given $U_0 : X \to \mathbb{R}$, we define

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0022-0396/$ – see front matter Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.jde.2013.12.018

$$U(t,x) := \sup\left\{ U_0\big(z(t)\big) - \int_0^t \left(\frac{1}{2}|z'|^2(r) - V\big(z(r)\big) \right) dr \colon z(0) = x,\ z(\cdot) \in AC\big([0,t];\mathsf{X}\big) \right\}.$$

Then U solves a Hamilton–Jacobi equation, formally written as

$$\partial_t U(t,x) = \frac{1}{2}\big|D_x U(t,x)\big|^2 + V(x), \tag{1.1}$$

where the *slope* (also called *local Lipschitz constant*) for a function $f : \mathsf{X} \mapsto \mathbb{R}$ is defined as

$$|Df|(x) := \limsup_{y\to x} \frac{|f(y) - f(x)|}{\mathsf{d}(y,x)}. \tag{1.2}$$

We are interested in a well-posedness theory for (1.1) and related equations. To fix the ideas and to separate difficulties of different nature, we do not pursue generality and only focus on the case of V with uniformly continuity in balls of finite radius, with possible growth to $-\infty$ at certain rate with respect to the metric distance, and uniformly bounded from above. The case $V = 0$ is of special interest, as the corresponding U then defines a Hopf–Lax semigroup which has applications to transportation inequalities in abstract metric space settings. Point-wise solution of (1.1) has been considered in Chapters 7 and 22 of Villani [30], in Section 3 of Ambrosio, Gigli and Savaré [3], Gozlan, Roberto and Samson [25]. The pointwise solution is sufficient for the purpose of the applications considered in these references. However, as is known in the case even when $\mathsf{X} \subset \mathbb{R}^d$, it is not good enough to ensure uniqueness for (1.1). Here we will generalize the notion of viscosity solution to metric space setting and develop a well-posedness theory.

The theory of viscosity solutions for (first order) Hamilton–Jacobi equation in infinite dimensions was initiated by Crandall and Lions [10,11]. One of the structural assumption is that X is a Hilbert space, or slightly more generally, certain Banach space with smooth norm. However, recent applications start to witness a situation where X is space of probability measures. This includes examples in statistical mechanics [4,17], optimal control and probability [14,15], classical game theory [8,9], fluid mechanics [18,19,13,16,20,21], and mean-field games (we refer to [7] for a compilation of references). In Sections 2 and 3 of this article, we extend the first order viscosity theory to general metric spaces setting by exploring maximum principles of the Hamiltonian operator. Other alternatives exist. For instance, [23,28] emphasize a formulation on path (hence the Lagrangian) by considering a sub-class of the Hamiltonians considered here. During the preparation of this article, we also received a preprint from the authors of [21] where their last section considers a related Cauchy problem using ideas of the same kind as in the first three sections of this article. Definition of viscosity solution is given for general Hamiltonians but well-posedness is treated for the case of $H(x,p) = H(p)$ and $H(x,p) = H(p) + f(x)$ only. Using Perron's method, solution is constructed implicitly. There is no convexity on p assumption on H. In this article, under a rather general structural assumption on H in (1.5), we treat Hamiltonians with much more general x-dependence. Growth estimate of solution is also provided. Our assumption implies that $p \mapsto H(x,p)$ is convex. However, the existence part of our well-posedness result is explicitly constructed using dynamical programming and value functions. Moreover, the proof of our uniqueness result does not critically rely on such convexity assumption. Finally, in Section 4, we give well-posedness for the resolvent equation relative to a special Hamiltonian in space of probability measures. Such issue, in the form of Cauchy problem, had been considered by [18, 21,26] but no well-posedness was given. In particular, the relation between the metric definition