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Stability of a critical nonlinear neutral delay differential equation

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Abstract

This work deals with a scalar nonlinear neutral delay differential equation issued from the study of wave propagation. A critical value of the coefficients is considered, where only few results are known. The difficulty follows from the fact that the spectrum of the linear operator is asymptotically closed to the imaginary axis. An analysis based on the energy method provides new results about the asymptotic stability of constant and periodic solutions. A complete analysis of the stability diagram is given in the linear homogeneous case. Under periodic forcing, existence of periodic solutions is discussed, involving a Diophantine condition on the period of the source.

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Keywords: Neutral delay differential equations; Energy method; Stability diagram; Periodic solutions; Small divisors; Non-exponential stability

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1. Introduction

In this work, we consider the following nonlinear neutral delay differential equation (NDDE)

$$\begin{cases} y'(t) + cy'(t-1) + f\big(y(t)\big) + g\big(y(t-1)\big) = s(t), & t > 0, \\ y(t) = y_0(t) \in H^1\big((-1,0), \mathbb{R}\big), & -1 < t < 0, \end{cases} \tag{1}$$

and we look for the solution $y \in H^1_{loc}((-1,+\infty), \mathbb{R})$ under the assumptions:

$$\begin{cases} c = \pm 1, & \text{(a)} \\ f \in C^1(\mathbb{R}, \mathbb{R}), \quad f' > 0, \quad f(0) = 0, & \text{(b)} \\ \exists \gamma < 1,\ \gamma \geqslant 0,\ \forall y \in \mathbb{R}, \quad \big|g(y)\big| \leqslant \gamma \big|f(y)\big|, & \text{(c)} \\ s(.) \text{ is a periodic function with period } T = \dfrac{2\pi}{\omega} > 0. & \text{(d)} \end{cases} \tag{2}$$

In some cases, additional assumptions will be required:

$$\begin{cases} \exists \eta < 1,\ \forall y \in \mathbb{R}, \quad \big|g'(y)\big| \leqslant \eta \big|f'(y)\big|, & \text{(a)} \\ \lim\limits_{y \to \pm\infty} f(y) = \pm\infty, & \text{(b)} \end{cases} \tag{3}$$

with $\eta \geqslant 0$. Eq. (1) is typically issued from hyperbolic partial differential equations with nonlinear boundary conditions [10,6]. A closely related system of neutral equations has been derived, for instance, in the case of elastic wave propagation across two nonlinear cracks [11]: y is then the dilatation of the crack, the shift 1 is the normalized travel time between the cracks, f and g denote the nonlinear contact law, and s is the T-periodic excitation.

A natural issue when dealing with (1) is to prove existence, uniqueness and stability of periodic solutions. Articles and reference books [14,5,3,15,9,7,13] usually address this question by considering a linear version of (1) with $|c| < 1$. The critical value $|c| = 1$ raises technical difficulties: the spectrum of the linearized operator is asymptotically closed to the left of the imaginary axis, hence exponential stability is lost and small divisors may be encountered.

An old contribution to the case $|c| = 1$ was done in [8]. In the linear case, algebraic rate of convergence was proven. In the nonlinear case with $c = -1$, algebraic convergence was also proven in this reference, but assuming small C^1 data. In both cases, the main tools involved are asymptotic expansions of characteristic roots, Laplace transforms and function series. Another recent contribution for linear degenerate retarded differential systems is given in [16].

The aim of our article is to push forwards the analysis of (1) in this critical case, especially to remove (when possible) the assumption of small solutions, and also to consider less regular initial data. A strategy based on energy analysis is followed. In the linear case, a stability analysis shows that the energy method is optimal. When the stability condition is violated, the existence of an essential instability is proven [10].

The paper is organized as follows. Section 2 treats the full nonlinear NDDE. The stability of the zero solution in the homogeneous case is proven (Theorem 1). Generalization to the non-homogeneous case with constant forcing is done (Corollaries 1 and 2). Non-constant periodic forcing is investigated in the case $g = 0$ (Proposition 1). The asymptotic stability of periodic solution is proven with the energy method but under a restrictive condition. In Section 3, new