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On the weak–strong uniqueness of Koch–Tataru's solution for the Navier–Stokes equations

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Abstract

We prove the weak–strong uniqueness between Koch–Tataru's solution and Leray's weak solution for three-dimensional incompressible Navier–Stokes equations.

Keywords: Weak–strong uniqueness; Koch–Tataru's solution; Leray's weak solution

1. Introduction

We consider three-dimensional incompressible Navier–Stokes equations

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(0) = u_0(x), \end{cases} \tag{1.1}$$

where $u = (u^1(t,x), u^2(t,x), u^3(t,x))$ and $p = p(t,x)$ denote the unknown velocity vector and the unknown scalar pressure of the fluid respectively, while $u_0(x)$ is a given initial velocity vector satisfying $\operatorname{div} u_0 = 0$.

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In a seminal paper [24], Leray constructed a global solution for any initial data in $L^2(\mathbb{R}^3)$ by using compactness method. More precisely, he proved that there exists a weak solution u of (1.1) satisfying

$$u \in L^\infty\big(0, T; L^2\big(\mathbb{R}^3\big)\big) \cap L^2\big(0, T; H^1\big(\mathbb{R}^3\big)\big) \quad \text{for any } T > 0, \tag{1.2}$$

and the energy inequality

$$\frac{1}{2}\big\|u(t)\big\|_{L^2}^2 + \int_0^t \big\|\nabla u(s)\big\|_{L^2}^2\, ds \leqslant \frac{1}{2}\|u_0\|_{L^2}^2; \tag{1.3}$$

And u satisfies (1.1) in the following sense

$$\int_{\mathbb{R}^3} u(t,x)\cdot\psi(t,x)\,dx - \int_{\mathbb{R}^3} u_0(x)\cdot\psi(0,x)\,dx$$
$$= -\int_0^t (\nabla u : \nabla\psi - u\otimes u : \nabla\psi - u\cdot\partial_t\psi)(s,x)\,ds \tag{1.4}$$

for any compactly supported, divergence-free vector field ψ. This is so-called Leray's weak solution or turbulent solution.

Later, there are a great deal of papers devoted to the study of regularity and uniqueness of weak solution. It is proved that Leray's weak solution u is regular if it satisfies

$$u \in L^q\big(0, T; L^p\big) \quad \text{with } \frac{2}{q} + \frac{3}{p} \leqslant 1,\ 3 \leqslant p \leqslant \infty; \tag{1.5}$$

On the other hand, if u and v are two Leray's weak solutions with the same initial data and one of solutions satisfies (1.5), then $u = v$. We refer to [11,12,20] and references therein. Recently, there are many papers devoted to refine the above results. We refer to [7,10,21,22] for the regularity of weak solution, and [8,15,14,23] for the uniqueness of weak solution. However, the regularity and uniqueness of weak solution still remain a challenging open problem without the additional conditions. For suitable weak solution, Caffarelli, Kohn and Nirenberg [4] proved that one-dimensional Hausdorff measure of possible singular set is zero. For the axisymmetric Navier–Stokes equations, Chen, Strain, Tsai and Yau [6] and Koch, Nadirashvili, Seregin and Šverák [19] proved the nonexistence of type I singularity by using different methods.

In 1964, another class of solutions were constructed by Fujita and Kato [13] for the initial data $u_0 \in \dot{H}^{\frac{1}{2}}$ by constructing a contraction map, thus are unique in their own class. Moreover, the solution is global in time if the $\dot{H}^{\frac{1}{2}}$ norm of u_0 is small enough. Later, Cannone, Meyer and Planchon [5] constructed the solution for the initial data $u_0 \in \dot{B}_{p,\infty}^{-1+\frac{3}{p}}$ for $3 < p < \infty$ (see the definitions in the next section); Koch and Tataru [18] proved the existence of solution for $u_0 \in BMO^{-1}$. Very recently, Germain [17], and Bourgain and Pavlović [3] proved the ill-posedness of (1.1) for the initial data $u_0 \in \dot{B}_{\infty,q}^{-1}$ for $q > 2$. So, the existence result of Koch and Tataru is