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J. Differential Equations 256 (2014) 2449–2479

Journal of Differential Equations

www.elsevier.com/locate/jde

Multiple solutions with precise sign for nonlinear parametric Robin problems

Nikolaos S. Papageorgiou [a], Vicenţiu D. Rădulescu [b,c,*]

[a] National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece
[b] King Abdulaziz University, Jeddah, Saudi Arabia
[c] Simion Stoilow Mathematics Institute of the Romanian Academy, Bucharest, Romania

Received 10 July 2013; revised 11 December 2013

Available online 24 January 2014

Abstract

We consider a parametric nonlinear Robin problem driven by the p-Laplacian. We show that if the parameter $\lambda > \hat{\lambda}_2 =$ the second eigenvalue of the Robin p-Laplacian, then the problem has at least three nontrivial solutions, two of constant sign and the third nodal. In the semilinear case ($p = 2$), we show that we can generate a second nodal solution. Our approach uses variational methods, truncation and perturbation techniques, and Morse theory. In the process we produce two useful remarks about the first two eigenvalues of the Robin p-Laplacian.

MSC: 35J20; 35J60; 58E05

Keywords: Nonlinear regularity; Nonlinear maximum principle; Robin p-Laplacian; Nodal and constant sign solutions; Extremal solutions; Morse theory

Contents

* Corresponding author.
E-mail addresses: npapg@math.ntua.gr (N.S. Papageorgiou), vicentiu.radulescu@cnrs.fr (V.D. Rădulescu).

http://dx.doi.org/10.1016/j.jde.2014.01.010

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric Robin problem:

$$\begin{cases} -\Delta_p u(z) = \lambda \left|u(z)\right|^{p-2} u(z) - f\left(z, u(z)\right) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p}(z) + \beta(z)\left|u(z)\right|^{p-2} u(z) = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

In this problem, Δ_p $(1 < p < \infty)$ denotes the p-Laplacian differential operator defined by

$$\Delta_p u = \operatorname{div}\left(\|Du\|^{p-2} Du\right) \quad \text{for all } u \in W^{1,p}(\Omega).$$

Also, $\frac{\partial u}{\partial n_p} = \|Du\|^{p-2}(Du, n)_{\mathbb{R}^N}$ with $n(z)$ being the outward unit normal at $z \in \partial\Omega$. In addition, $\lambda > 0$ is a parameter and $f(z,x)$ is a Carathéodory perturbation (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z,x)$ is continuous), which exhibits $(p-1)$-superlinear growth near $\pm\infty$.

Our aim in this paper is to prove a multiplicity theorem for problem (P_λ) for all $\lambda > 0$ big. More precisely, we show that, if $\hat{\lambda}_2$ is the second eigenvalue of $-\Delta_p$ with Robin boundary conditions (denoted by $-\Delta_p^R$) and $\lambda > \hat{\lambda}_2$ then problem (P_λ) admits at least three nontrivial solutions, two of constant sign (the first positive and the second negative) and the third solution is nodal (sign changing). Moreover, in the semilinear case $(p = 2)$, we show the existence of a second nodal solution, for a total of four nontrivial solutions all with sign information. Our approach uses variational methods coupled with suitable truncation and perturbation techniques and Morse theory.

This kind of problem was studied for semilinear (that is, $p = 2$) Dirichlet equations by Ambrosetti and Lupo [2], Ambrosetti and Mancini [3] and Struwe [20], [21, p. 133]. Extensions to Dirichlet p-Laplacian equations can be found in Papageorgiou and Papageorgiou [18]. However, none of the aforementioned works produced nodal solutions and the hypotheses on the data of the problem are more restrictive. Another class of Robin eigenvalue problems was investigated by Duchateau [7], who proved multiplicity results producing two solutions with no sign information.

2. Mathematical background – auxiliary results

Let X be a Banach space and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$, we say that φ satisfies the Palais–Smale condition (PS-condition for short), if the following is true

> "Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\left\{\varphi(x_n)\right\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and
>
> $\varphi'(x_n) \to 0$ in X^*admits a strongly convergent subsequence."