Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 256 (2014) 2480–2501

Journal of Differential Equations

www.elsevier.com/locate/jde

The exterior Dirichlet problem for fully nonlinear elliptic equations related to the eigenvalues of the Hessian

Haigang Li *, Jiguang Bao

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China

Received 25 March 2013

Available online 22 January 2014

Abstract

In this paper, we establish the existence theorem for the exterior Dirichlet problems for a class of fully nonlinear elliptic equations, which are related to the eigenvalues of the Hessian matrix, with prescribed asymptotic behavior at infinity. This extends the previous results on Monge–Ampère equation and k-Hessian equation to more general cases, in particular, including the special Lagrangian equation.

1. Introduction

In this paper, we study the existence of viscosity solutions to exterior Dirichlet problem for the following fully nonlinear, second order partial differential equation of the form

$$\begin{cases} f\big(\lambda\big(D^2u\big)\big)=1, & \text{in } \mathbb{R}^n\setminus\overline{D}, \quad \text{(a)}\\ u=\varphi, & \text{on } \partial D, \quad \text{(b)} \end{cases} \tag{1.1}$$

where D is a bounded open set in $\mathbb{R}^n$ $(n\geqslant 3)$, $f(\lambda)$ is a given smooth symmetric function of the eigenvalues $\lambda=(\lambda_1,\dots,\lambda_n)$ of the Hessian matrix D^2u. The typical cases of f include the elementary symmetric functions

* Corresponding author.
E-mail addresses: hgli@bnu.edu.cn (H. Li), jgbao@bnu.edu.cn (J. Bao).

0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2014.01.011

$$\sigma_k(\lambda) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k = 1, \ldots, n, \tag{1.2}$$

the quotients of elementary symmetric functions,

$$\sigma_{k,l}(\lambda) = \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}, \quad 1 \leqslant l < k \leqslant n, \tag{1.3}$$

and the special Lagrangian operator

$$\sum_{i=1}^{n} \arctan \lambda_i.$$

The elementary symmetric functions (1.2) are embraced by [6] and treated as well by Ivochkina [20]. Note that the case $k = 1$ corresponds to Laplace operator, while for $k = n$, we have the classical Monge–Ampère operator.

In bounded domains $\Omega \subset \mathbb{R}^n$, Caffarelli, Nirenberg and Spruck treated the traditional (or interior) Dirichlet problem in [6],

$$\begin{cases} f\big(\lambda\big(D^2u\big)\big) = \psi(x), & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where they demonstrated the existence of classical solutions, under various hypothesis on the function f and the domain Ω. The results in [6] extended their previous work [5], and that of Krylov [25], Ivochkina [19] and others on equations of Monge–Ampère type,

$$\det D^2 u = \psi(x), \tag{1.5}$$

where ψ is a given function in $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Trudinger [28] provided a new method to obtain the double normal second derivative estimation and extended the result in [6] to the important examples of quotients of elementary symmetric functions (1.3), which do not satisfy the structure hypothesis on f in [6]. More results in a bounded domain on these types of equations can be referred to Trudinger [29], Urbas [31] and the references therein.

In contrast to numerous results on the traditional Dirichlet problems (1.4) in bounded domains, less is known about the exterior Dirichlet problems (1.1a)–(1.1b) where the domain is unbounded. Especially, in the whole space $\mathbb{R}^n$, a classical theorem of Jörgens [23], Calabi [7], and Pogorelov [27] states that any classical convex solution of

$$\det\big(D^2u\big) = 1, \quad \text{in } \mathbb{R}^n \tag{1.6}$$

must be a quadratic polynomial. More extensive and outstanding results on (1.6) are given by Cheng and Yau [9], Caffarelli [3], Jost and Xin [24], Trudinger and Wang [30] and many others. Caffarelli and Li [4] extended the Jörgens–Calabi–Pogorelov theorem to exterior domains, namely that if u is a locally convex viscosity solution of

$$\det\big(D^2u\big) = 1, \quad \text{in } \mathbb{R}^n \setminus \overline{D} \tag{1.7}$$