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Bounded solutions for a forced bounded oscillator without friction

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Abstract

Under the validity of a Landesman–Lazer type condition, we prove the existence of solutions bounded on the real line, together with their first derivatives, for some second order nonlinear differential equation of the form $\ddot{u} + g(u) = p(t)$, where the reaction term g is bounded. The proof is variational, and relies on a dual version of the Nehari method for the existence of oscillating solutions to superlinear equations.

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1. Introduction

This paper concerns the existence of solutions, bounded on the real line together with their first derivative, for the differential equation

$$\ddot{u} + g(u) = p(t), \tag{1}$$

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where $g \in \mathcal{C}^2(\mathbb{R})$ is bounded, increasing, and has exactly one inflection point, and $p \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ admits asymptotic average $A(p) \in \mathbb{R}$, that is

$$\lim_{T\to+\infty} \frac{1}{T} \int_t^{t+T} p(s)\,ds = A(p),$$

uniformly in $t \in \mathbb{R}$. Such an equation describes the forced motions of an oscillator exhibiting saturation effects. As a model problem, the reader may think to the equation

$$\ddot{u} + \arctan u = p(t),$$

even though we do not require any symmetry assumption on the reaction term g. Under the above assumptions, the main result we prove is the following theorem.

Theorem 1.1. *Eq. (1) admits a bounded solution if and only if*

$$g(-\infty) < A(p) < g(+\infty). \tag{2}$$

In such a case, Eq. (1) admits a countable set of bounded solutions, having arbitrarily large L^∞-norm.

The motivation for our investigation relies on the papers [1,6], which in turn have been inspired by some classical results of Landesman–Lazer type holding in the periodic framework. Such studies concern the equation

$$\ddot{u} + c\dot{u} + g(u) = p(t), \tag{3}$$

where $c \in \mathbb{R}$ and the continuous function g, not necessarily monotone, admits limits at $\pm\infty$, with the property that

$$g(-\infty) < g(s) < g(+\infty)$$

for every s. Also the cases $g(\pm\infty) = \pm\infty$ can be considered, requiring g to be sublinear at infinity if $c = 0$. When p is T-periodic, it is nowadays well known that Eq. (1) admits a periodic solution if and only if the Landesman–Lazer condition

$$g(-\infty) < \frac{1}{T} \int_0^T p(s)\,ds < g(+\infty)$$

is satisfied, regardless of the constant c; this result was first proved by Lazer, using the Schauder fixed point theorem, see [4]. When p is merely bounded, one would like to find analogous conditions for the search of bounded solutions. This problem was first studied by Ahmad [1], under the assumption that p has asymptotic average, in the sense explained above; by means of techniques of the qualitative theory of dissipative equations, the existence of a bounded solution is characterized, whenever $c \neq 0$, by (2). The case in which p is an arbitrary continuous function