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Weak solutions of the Navier–Stokes equations with non-zero boundary values in an exterior domain satisfying the strong energy inequality

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Abstract

In an exterior domain $\Omega \subset \mathbb{R}^3$ and a time interval $[0, T)$, $0 < T \leqslant \infty$, consider the instationary Navier–Stokes equations with initial value $u_0 \in L^2_\sigma(\Omega)$ and external force $f = \operatorname{div} F$, $F \in L^2(0, T; L^2(\Omega))$. As is well-known there exists at least one weak solution in the sense of J. Leray and E. Hopf with vanishing boundary values satisfying the strong energy inequality. In this paper, we extend the class of global in time Leray–Hopf weak solutions to the case when $u_{|\partial\Omega} = g$ with non-zero time-dependent boundary values g. Although uniqueness for these solutions cannot be proved, we show the existence of at least one weak solution satisfying the strong energy inequality and a related energy estimate.

MSC: 35Q30; 35J65; 76D05

Keywords: Instationary Navier–Stokes equations; Weak solutions; Strong energy inequality; Non-zero boundary values; Time-dependent data; Exterior domain

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary of class $C^{1,1}$, and let $[0, T)$, $0 < T \leqslant \infty$, be a time interval. In $\Omega \times [0, T)$ we consider the instationary Navier–Stokes system with viscosity $\nu > 0$ and data f, g, u_0 in the form

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$$\begin{aligned} u_t - \nu\Delta u + u \cdot \nabla u + \nabla p = f, \qquad \operatorname{div} u = 0, \\ u|_{\partial\Omega} = g, \qquad u(0) = u_0. \end{aligned} \tag{1.1}$$

For the data we assume the following:

$$\begin{aligned} f = \operatorname{div} F, \quad F \in L^2\big(0,T; L^2(\Omega)\big),\ u_0 \in L^2_\sigma(\Omega), \\ g \in L^4\big(0,T; W^{-\frac{1}{4},4}(\partial\Omega)\big) \cap L^{s_0}\big(0,T; W^{-\frac{1}{q_0},q_0}(\partial\Omega)\big), \\ \frac{2}{s_0} + \frac{3}{q_0} = 1, \quad 2 < s_0 < \infty,\ 3 < q_0 < \infty; \end{aligned} \tag{1.2}$$

the initial data u_0 has to satisfy further assumptions to be introduced later, see Section 5.

A weak solution u to (1.1) will be constructed in the form $u = v + E$ where E solves an instationary Stokes system with the boundary data g, and v solves a type of Navier–Stokes system with additional perturbation terms related to E but homogeneous Dirichlet data on $\partial\Omega$. Therefore, the problem splits into two almost independent parts, the construction of E as weak (or very weak) solution of a Stokes system and the analysis of a perturbed Navier–Stokes system. It is worth mentioning that the second step needs only very low assumptions on E known from the theory of very weak solutions (E lies in Serrin's class $L^{s_0}(0,T; L^{q_0}(\Omega))$) and from the classical theorem for weak solutions to satisfy the energy identity ($E \in L^4(0,T; L^4(\Omega))$); here the assumptions on g and on u_0 in (1.2) are not explicitly needed.

To be more precise, we have to find first of all a (so-called) very weak solution of the inhomogeneous Stokes system

$$\begin{aligned} E_t - \nu\Delta E + \nabla h = f_0, \qquad \operatorname{div} E = 0, \\ E|_{\partial\Omega} = g, \qquad E(0) = E_0, \end{aligned} \tag{1.3}$$

see [2–6] and, for the case of exterior domains, [7], in $\Omega \times [0,T)$ with suitable data $f_0 = \operatorname{div} F_0$ and E_0; here ∇h means the associated pressure. At first sight, it seems to suffice to choose $f_0 = 0$, $F_0 = 0$, but for later application it will be helpful to consider general data f_0, F_0, see Assumption 1.6 to be used in Corollary 1.7 below. Setting

$$v = u - E, \qquad \tilde p = p - h, \qquad f_1 = f - f_0, \qquad v_0 = u_0 - E_0 \tag{1.4}$$

we write (1.1) as a perturbed Navier–Stokes system with homogeneous boundary data $v|_{\partial\Omega} = 0$

$$\begin{aligned} v_t - \nu\Delta v + (v+E) \cdot \nabla(v+E) + \nabla\tilde p = f_1, \qquad \operatorname{div} v = 0, \\ v|_{\partial\Omega} = 0, \qquad v(0) = v_0 \end{aligned} \tag{1.5}$$

with the new perturbation terms

$$(v+E) \cdot \nabla(v+E) = \operatorname{div}\big(v \otimes v + (E \otimes v + v \otimes E) + E \otimes E\big);$$

here $E \otimes v = (E_i v_j)_{i,j=1,2,3}$ denotes the dyadic product of the vector fields E and v and the divergence is taken columnwise, i.e., $\operatorname{div} E \otimes v = (\sum_{i=1}^3 \partial_i(E_i v_j))_{j=1,2,3}$ ($= E \cdot \nabla v$, since $\operatorname{div} E = 0$).