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# Asymptotic behaviour of travelling waves for the delayed Fisher–KPP equation ☆

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**Abstract**

In this work we study the behaviour of travelling wave solutions for the diffusive Hutchinson equation with time delay. Using a phase plane analysis we prove the existence of travelling wave solution for each wave speed $c \geqslant 2$. We show that for each given and admissible wave speed, such travelling wave solutions converge to a unique maximal wavetrain. As a consequence the existence of a nontrivial maximal wavetrain is equivalent to the existence of travelling wave solution non-converging to the stationary state $u = 1$.


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## 1. Introduction

The aim of this article is to study the entire bounded and positive orbits of the following second order delay differential equation:

$$-u''(z) + cu'(z) = u(z)\big(1 - u(z-h)\big) \quad \text{for } z \in \mathbb{R}, \tag{1.1}$$

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with $c > 0$ and $h > 0$, together with the conditions at infinity

$$\lim_{z\to-\infty} u(z) = 0 \quad \text{and} \quad \liminf_{z\to\infty} u(z) > 0. \tag{1.2}$$

The above problem arises when looking at travelling wave solutions with speed $c$ of the so-called Hutchinson equation, also referred to as the diffusive delayed logistic equation, which reads:

$$(\partial_t - \Delta)U(t,x) = U(t,x)\big[1 - U(t-\tau,x)\big], \quad t > 0,\ x \in \mathbb{R}^N. \tag{1.3}$$

A travelling wave solution with speed $c$ in the unit direction $e \in \mathbb{S}^{N-1}$ for the above equation is an entire solution of the form

$$U(t,x) \equiv u(z) \quad \text{with} \quad z = xe + ct,$$

so that the profile $u$ satisfies (1.1) with $h = c\tau$.

When $h = 0$, that is $\tau = 0$, we recover the classical Fisher–KPP equation. It is known since the pioneering works of Kolmogorov, Petrovsky and Piskunov [16] and Fisher [9] in the 30s that for all $c \geqslant 2$, Eqs. (1.1)–(1.2) with $h = 0$ admit a travelling wave solution $u$, which is increasing and converges to 1 at $+\infty$. The travelling wave with minimal speed $c = 2$ attracts, in a sense, the solutions of initial value problems associated with compactly supported initial data. Hence, such solutions model population invasion processes.

The introduction of delayed or nonlocal effects in reaction–diffusion equations is known to give rise to nontrivial periodic steady states since the pioneering paper of Turing [24]. The equation

$$-u'' + cu' = u(1 - \phi \star u), \tag{1.4}$$

where $\phi$ is an even probability distribution, has been introduced in [5,12,10] in an evolutionary dynamics framework. Nontrivial periodic steady states could then be interpreted as the emergence of new species. The existence of waves for such equations has been investigated by Berestycki, Perthame, Ryzhik and the second author in [3]. The convergence of such waves to 1 at $+\infty$ is unclear, which lead these authors to introduce a generalized notion of travelling waves, that we now adapt to Eq. (1.1).

**Definition 1.1.** (See [3].) We say that a positive solution $u \in \mathcal{C}^2(\mathbb{R})$ of (1.1) is a *travelling wave* (of speed $c > 0$) if it is bounded, it converges to 0 at $-\infty$ and $\liminf_{z\to+\infty} u(z) > 0$. In other words, a travelling wave is a solution of:

$$\begin{cases} -u''(z) + cu'(z) = u(z)\big(1 - u(z-h)\big) & \text{in } \mathbb{R}, \\ u \text{ is positive and bounded over } \mathbb{R}, \\ u(-\infty) = 0, \qquad \liminf_{z\to+\infty} u(z) > 0. \end{cases} \tag{1.5}$$

The existence of such waves for the nonlocal equation (1.4) was proved in [3] for all $c \geqslant 2$. The monotonicity of such waves was completely characterized in [7]. Numerics indicate that such waves might always converge to 1 [21], but this conjecture is only proved for large speeds [1]