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Mass transport problems for the Euclidean distance obtained as limits of p-Laplacian type problems with obstacles

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Abstract

In this paper we analyze a mass transportation problem that consists in moving optimally (paying a transport cost given by the Euclidean distance) an amount of a commodity larger than or equal to a fixed one to fulfil a demand also larger than or equal to a fixed one, with the obligation of paying an extra cost of $-g_1(x)$ for extra production of one unit at location x and an extra cost of $g_2(y)$ for creating one unit of demand at y. The extra amounts of mass (commodity/demand) are unknowns of the problem. Our approach to this problem is by taking the limit as $p \to \infty$ to a double obstacle problem (with obstacles g_1, g_2) for the p-Laplacian. In fact, under a certain natural constraint on the extra costs (that is equivalent to impose that the total optimal cost is bounded) we prove that this limit gives the extra material and extra demand needed for optimality and a Kantorovich potential for the mass transport problem involved. We also show that this problem can be interpreted as an optimal mass transport problem in which one can make the transport directly (paying a cost given by the Euclidean distance) or may hire a courier that cost $g_2(y) - g_1(x)$ to pick up a unit of mass at y and deliver it to x. For this different interpretation we provide examples and a decomposition of the optimal transport plan that shows when we have to use the courier.

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1. Introduction

Our main goal in this paper is to show that the limit as $p \to \infty$ for the double obstacle problem for the p-Laplacian gives a complete answer to an optimal mass transport problem with the Euclidean distance.

Consider the following variational problem where a double obstacle is considered,

$$\inf_{\substack{u\in W^{1,p}(\Omega):\\ g_1\leqslant u\leqslant g_2 \text{ in } \Omega}} \int_\Omega \frac{|\nabla u(x)|^p}{p}\,dx - \int_\Omega f(x)u(x)\,dx. \tag{1.1}$$

We show that, provided that the restriction set is not empty, the obstacle problem has a solution for every fixed $p > N$ and, in addition, under a natural Lipschitz-type constraint on the obstacles, we prove that there is a uniform limit (along subsequences) as $p \to \infty$, u_∞, that is a solution of the variational problem

$$\max_{\substack{w\in W^{1,\infty}(\Omega):\\ \|\nabla w\|_{L^\infty(\Omega)}\leqslant 1,\\ g_1\leqslant w\leqslant g_2 \text{ in } \Omega}} \int_\Omega w(x)f(x)\,dx. \tag{1.2}$$

As we have mentioned, our main aim is to relate this optimization problem with an adequate optimal transport problem.

Limits as $p \to \infty$ of similar type problems are related to optimal mass transport problems for the Euclidean distance. In fact, this relation was the key to the first complete proof of the existence of an optimal transport map for the classical Monge problem given by Evans and Gangbo in [7]. See also [8], where a problem with import/export taxes was studied (the present work contains and extends some of the results proved there), and [9], where an optimal matching problem is analyzed. Note that the usual Euclidean distance is not a strictly convex cost. This makes this optimal mass transport different from the strictly convex cost case in which there is existence of a convex function (solution to a Monge Ampere type problem) whose gradient provides an optimal transport map, see [11]. For notation and general results on Mass Transport Theory we refer to [1,3,6,7,11,12].

We are going to show that the limit variational problem (1.2) is related to the following optimal transport problem (see the next section for a precise mathematical formulation):

An optimal mass transport problem with taxes. Assume that we have some production in a domain Ω encoded in f_+ and some consumption encoded in f_-. We have the right to enlarge our previous production f_+, over all the domain included the boundary, paying an extra cost given by $-g_1(x)$ for each extra unit that we can produce at x, and we can create new demand paying an extra cost given by $g_2(y)$ for each extra unit of demand created at y (for example, this can come from advertising). Our main goal is to move the whole production and satisfy the whole demand minimizing the total cost of the operation. To transport one unit of material from x to y we pay as transport cost the Euclidean distance $|x - y|$. We will prove that solutions to the p-Laplacian type problem associated with (1.1) give an approximation to the extra production/demand necessary in the process and to a Kantorovich potential for the corresponding transport problem.