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Inverse Jacobian multipliers and Hopf bifurcation on center manifolds

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Abstract

In this paper we consider a class of higher dimensional differential systems in $\mathbb{R}^n$ which have a two dimensional center manifold at the origin with a pair of pure imaginary eigenvalues. First we characterize the existence of either analytic or C^∞ inverse Jacobian multipliers of the systems around the origin, which is either a center or a focus on the center manifold. Later we study the cyclicity of the system at the origin through Hopf bifurcation by using the vanishing multiplicity of the inverse Jacobian multiplier.

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Keywords: Inverse Jacobian multiplier; Hopf bifurcation; Normal form; Center; Focus

1. Background and statement of the main results

For real planar differential systems, the problems on center–focus and Hopf bifurcation are classical and related. They are important subjects in the bifurcation theory and also in the study of Hilbert's 16th problem [6,7,15,17].

For planar non-degenerate center, Poincaré provided an equivalent characterization.

Poincaré center Theorem. *For a real planar analytic differential system with the origin as a singularity having a pair of pure imaginary eigenvalues, the origin is a center if and only if the*

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system has a local analytic first integral, and if and only if the system is analytically equivalent to

$$\dot{u} = -\mathbf{i}u\big(1 + g(uv)\big), \qquad \dot{v} = \mathbf{i}v\big(1 + g(uv)\big),$$

with $g(uv)$ without constant terms, where we have used the conjugate complex coordinates instead of the two real ones.

This result has a higher dimensional version, see for instance [13,18,20], which characterizes the equivalence between the analytic integrability and the existence of analytic normalization of analytic differential systems to its Poincaré–Dulac normal form of a special type.

Reeb [14] in 1952 provided another characterization on planar centers via inverse integrating factor. Recall that a function V is an *inverse integrating factor* of a planar differential system if $1/V$ is an integrating factor of the system. From [8,9,11,12] we know that inverse integrating factors have better properties than integrating factors.

Reeb center Theorem. *Real planar analytic differential system*

$$\dot{x} = -y + f_1(x, y), \qquad \dot{y} = x + f_2(x, y),$$

has the origin as a center if and only if it admits a real analytic local inverse integrating factor with non-vanishing constant part.

Poincaré center Theorem was extended to higher dimensional differential systems which have a two dimensional center manifold by Lyapunov. Consider analytic differential systems in $\mathbb{R}^n$

$$\begin{aligned}
\dot{x} &= -y + f_1(x, y, z) = F_1(x, y, z),\\
\dot{y} &= x + f_2(x, y, z) = F_2(x, y, z),\\
\dot{z} &= Az + f(x, y, z) = F(x, y, z),
\end{aligned} \tag{1.1}$$

with $z = (z_3, \dots, z_n)^{tr}$, A is a real square matrix of order $n-2$, and $f = (f_3, \dots, f_n)^{tr}$ and $F = (F_3, \dots, F_n)^{tr}$. Hereafter we use *tr* to denote the transpose of a matrix. Moreover we assume that $\mathbf{f} := (f_1, f_2, f) = O(|(x, y, z)|^2)$ are n dimensional vector valued analytic functions. We denote by

$$\mathcal{X} = F_1(x, y, z)\frac{\partial}{\partial x} + F_2(x, y, z)\frac{\partial}{\partial y} + \sum_{j=3}^{n} F_j(x, y, z)\frac{\partial}{\partial z_j}$$

the vector field associated to systems (1.1).

Assume that the eigenvalues of A all have non-zero real parts. Then from the Center Manifold Theorem we get that system (1.1) has a center manifold tangent to the (x, y) plane at the origin (of course center manifolds are not necessary unique, and may not be analytic even not C^∞). Moreover this center manifold can be represented as

$$\mathcal{M}^c = \bigcap_{j=3}^{n} \big\{z_j = h_j(x, y)\big\}. \tag{1.2}$$