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Blowup issues for a class of nonlinear dispersive wave equations ☆

Lorenzo Brandolese *, Manuel Fernando Cortez

Université de Lyon, Université Lyon 1, CNRS UMR 5208 Institut Camille Jordan, 43 bd. du 11 novembre, Villeurbanne Cedex F-69622, France

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Abstract

In this paper we consider the nonlinear dispersive wave equation on the real line, $u_t - u_{txx} + [f(u)]_x - [f(u)]_{xxx} + [g(u) + \frac{f''(u)}{2}u_x^2]_x = 0$, that for appropriate choices of the functions f and g includes well known models, such as Dai's equation for the study of vibrations inside elastic rods or the Camassa–Holm equation modelling water wave propagation in shallow water. We establish a *local-in-space* blowup criterion (*i.e.*, a criterion involving only the properties of the data u_0 in a neighborhood of a *single point*) simplifying and extending earlier blowup criteria for this equation. Our arguments apply both to the finite and infinite energy cases, yielding the finite time blowup of strong solutions with possibly different behavior as $x \to +\infty$ and $x \to -\infty$.

Keywords: Rod equation; Camassa–Holm; Shallow water; Wave breaking

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* Corresponding author.
E-mail addresses: brandolese@math.univ-lyon1.fr (L. Brandolese), cortez@math.univ-lyon1.fr (M.F. Cortez).
URL: http://math.univ-lyon1.fr/~brandolese (L. Brandolese).

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1. Introduction

The experimental observation by the naval architect Scott Russel of solitary waves propagating in channels at different speeds, and interacting in a nonlinear way before recovering their initial shape, motivated the studies on the mathematical modelling of water wave motion at the end of the XIX century. The first works can be retraced back to Boussinesq, Rayleigh, Korteweg and de Vries. The celebrated KdV equation allows for a first mathematical description of such phenomena. This equation can be derived as an asymptotic model from the free surface Euler equations in the so-called shallow water regime $\mu = h^2/\lambda^2 \ll 1$, where h and λ denote respectively the average elevation of the liquid over the bottom and the characteristic wavelengt. It models small amplitude waves, *i.e.* waves such that the dimensionless amplitude parameter $\epsilon = a/h$ satisfies $\epsilon = O(\mu)$, where a is the typical amplitude.

Such small amplitude waves feature both nonlinear and dispersive effects. For larger amplitude waves nonlinear effects become preponderant and wave breaking can eventually occur. As the KdV equation is no longer suitable for the description of breaking mechanisms — its solutions remain smooth for all time — a considerable effort was made toward the modelling of larger amplitude, possibly breaking waves, see, *e.g.*, the monograph [37]. Such studies culminated with the derivation in 1993, by Camassa and Holm, [8,9] of an equation obtained from the vertically averaged water wave system, written in Lie–Poisson Hamiltonian form, by an asymptotic expansion preserving the Hamiltonian structure. The scaling of validity of the Camassa–Holm equation is $\mu \ll 1$ and $\epsilon = O(\sqrt{\mu})$: such scaling includes that of KdV allowing higher order accuracy. Alternative rigorous derivations of the Camassa–Holm equation are also available, see [14,28]. Such equation attracted a considerable interest in the past 20 years, not only due its hydrodynamical relevance (it was the first equation capturing both soliton-type solitary waves as well as breaking waves) but also because of its extremely rich mathematical structure. In fact, the Camassa–Holm equation was written for the first time in a different context, as one of the 12 integrable equations classified by Fokas and Fuchssteiner [21] and obtained from a nonlinear operator satisfying suitable defining properties, applying a recursive operator that is a hereditary symmetry.

The Camassa–Holm equation is usually written as

$$u_t + \kappa u_x - u_{xxt} + 3uu_x = uu_{xxx} + 2u_x u_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

where u can be interpreted as a horizontal velocity of the water at a certain depth and κ is the dispersion parameter.

The dispersionless case $\kappa = 0$ is of mathematical interest as in this case the equation possess soliton solutions (often named *peakons*) peaked at their crest, of the form $u_c(t,x) = ce^{-|x-ct|}$. Multi-peakon interactions are studied in [8,9]. For $\kappa > 0$ the equation admits smooth solitons.

In the shallow water interpretation, however, κ is proportional to the square root of the water depth and cannot be zero. On the other hand, the same equation, with $\kappa = 0$ appears, *e.g.*, in the study of the dynamics of a class of non-Newtonian, second-grade fluids (see [7]), or when modelling vibrations inside hyper-elastic rods. In the latter case peakons correspond to physical solutions. More in general, the propagation of nonlinear waves inside cylindrical hyper-elastic rods, assuming that the diameter is small when compared to the axial length scale, is described by the one dimensional equation (see [18]),

$$v_\tau + \sigma_1 v v_\xi + \sigma_2 v_{\xi\xi\tau} + \sigma_3(2v_\xi v_{\xi\xi} + v v_{\xi\xi\xi}) = 0, \quad \xi \in \mathbb{R},\ \tau > 0.$$