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Instability of edge waves along a sloping beach

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Abstract

The stability of three-dimensional edge waves along a sloping beach described in the Lagrangian framework is investigated by the theory of short-wavelength perturbations. We prove that the edge waves with the steepness parameter higher than $\frac{7}{18}\sin\alpha$, α being the sloping angle of the beach, are unstable.

Keywords: Edge waves; Lagrangian framework; Short-wavelength method; Localized instability analysis

1. Introduction

Edge waves are waves which propagate parallel to the shoreline, their crests point offshore and their amplitude is maximum at the shore and decays asymptotically to zero in the offshore direction. Mathematically, the edge waves are modelled by the Euler equations with appropriate kinematic and dynamic boundary conditions. Within the framework of linear theory, the governing equations are linearized and this simplification permits a thorough analysis. In 1846, Stokes [34] presented an exact single-mode solution of the linearized problem which propagates in the longshore direction, and whose amplitude decays exponentially in the offshore direction (see also [25, §240], [24]). In 1952, Ursell [36] demonstrated that more linear edge-wave modes appear as the constant slope of the beach decreases and he gave the first complete solution of the linearized problem (being a mixture of continuous and discrete spectra) (see also [37,12,24]). Whitham [37] gave nonlinear corrections to the solution of Ursell and showed how edge waves

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can be determined systematically. Many other authors have made important contributions to the theory of edge waves, we refer the reader to the surveys [38,12,24]. The edge waves play an important role in a number of near-shore processes; a good source of discussions and references in relation to the physical importance of edge-waves is [22]. Interesting studies within the class of edge waves are also [21,35,32].

In 1966, Yih [39] demonstrated that a coordinate transformation of Gerstner's solution, this is an explicit solution to the full water-wave problem for infinite depth (see [17,4,19]), produces an edge-wave solution propagating along a sloping beach. A similar observation is given by Mollo-Christensen [33]. However, their treatment provides only an implicit form for the free water surface. In 2001, Constantin [5] showed that Gerstner's solution can be used to construct an explicit edge-wave solution to the full water-wave problem propagating along a sloping beach. A detailed analysis of the edge-wave dynamics as well as examples of the surface profiles and of the run-up patterns, are readily obtained by using the Lagrangian approach to the water motion, that is, by considering the paths followed by each fixed fluid particle. In Section 2 we present this explicit edge-wave solution due to Constantin [5].

Once an exact solution is available, the stability issue becomes important. The question of stability or instability is the question of whether there exist perturbations that grow as time progresses. Beside the classical spectral and energy methods, the short-wavelength method (analogous to geometrical optics method from the theory of light rays) plays an important role in the hydrodynamic stability/instability theory of the three-dimensional inviscid incompressible fluids. This method, which was developed independently by Bayly [1], Friedlander and Vishik [14] and Lifschitz and Hameiri [30] (see also the surveys by Friedlander and Yudovich [16], Friedlander and Lipton-Lifschitz [13]), considers the evolution of wave packet perturbations to the flow that can be analyzed in the short-wavelength limit. Along the flow trajectories, the wave packet dynamics is described by a system of ordinary differential equations. In the language of geometrical optics, this system of ordinary differential equations consists of the eikonal equation for the wave vector and the transport equation for the velocity amplitude. The asymptotic behaviour of the velocity amplitude characterizes stability. A sufficient condition for the instability of the flow is as the magnitude of the amplitude of the perturbed velocity grows in time without bound along at least one trajectory. In Section 3 we present the short-wavelength instability method.

The short-wavelength method has been applied to a number of classical stability problems: for example, the stability of Kirchoff–Kida vortices [3], two-dimensional and quasi-two dimensional flows [2], vortex rings with and without swirl [28,31]. The method was successfully applied when the basic flow is described in the Lagrangian framework: see Leblanc [26] for Gerstner's waves and Constantin and Germain [10] for the equatorially trapped waves [8]. In Section 4 we show that this method can be applied to edge waves along a sloping beach too. We prove that the edge waves with the steepness parameter higher than $\frac{7}{18}\sin\alpha$, α being the sloping angle of the beach, are unstable.

We mention the potential applicability of the short-wavelength instability method to other explicit Lagrangian solutions [35,32,9,20].

2. Description of the edge waves along a sloping beach

We consider a sloping beach with slope angle $\alpha \in (0, \frac{\pi}{2})$. We choose the cartesian axes such that $z = 0$, $y \leqslant b_0 \leqslant 0$ is the sloping bed, $z = (b_0 - y)\tan\alpha$, $y \leqslant b_0 \leqslant 0$ is the horizontal plane of the undisturbed water surface, the x-axis is taken along the shoreline and z is measured vertically